In this chapter

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Integration techniques and applications

You will have seen in your Maths Methods course and elsewhere that some functions can be antidifferentiated (integrated) using standard rules. These common results are shown in the table below where the function \( f(x) \) has an antiderivative \( F(x) \).

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ax^n )</td>
<td>( \frac{ax^{n+1}}{n+1} + c )</td>
</tr>
<tr>
<td>( \frac{1}{x} )</td>
<td>( \log_e kx + c )</td>
</tr>
<tr>
<td>( e^{kx} )</td>
<td>( \frac{e^{kx}}{k} + c )</td>
</tr>
<tr>
<td>( \sin kx )</td>
<td>( -\frac{\cos kx}{k} + c )</td>
</tr>
<tr>
<td>( \cos kx )</td>
<td>( \frac{\sin kx}{k} + c )</td>
</tr>
<tr>
<td>( \sec^2 kx )</td>
<td>( \frac{\tan kx}{k} + c )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{a^2 - x^2}}, x \in (-a, a) )</td>
<td>( \sin^{-1} \frac{x}{a} + c )</td>
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<tr>
<td>( \frac{-1}{\sqrt{a^2 - x^2}}, x \in (-a, a) )</td>
<td>( \cos^{-1} \frac{x}{a} + c )</td>
</tr>
<tr>
<td>( \frac{a}{a^2 + x^2} )</td>
<td>( \tan^{-1} \frac{x}{a} + c )</td>
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In this chapter you will learn how to find antiderivatives of more complex functions using various techniques.

**Technique 1: Substitution where the derivative is present in the integrand**

Since \( \frac{d}{dx} [f(x)]^{n+1} = (n + 1) f'(x) [f(x)]^n \), \( n \neq -1 \), as an application of the chain rule, then it follows that: \( \int f'(x) [f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1 \).

Since \( \frac{d}{dx} \log_e f(x) = \frac{f'(x)}{f(x)} ; f(x) \neq 0 \) then it follows that \( \int \frac{f'(x)}{f(x)} \, dx = \log_e f(x) + c \).

The method relies on the derivative, or multiple of the derivative, being present and recognisable. Then, the appropriate substitutions may be made according to the above rules.
Find the antiderivative of the following expressions.

a \((x + 3)^7\)  
b \(4x(2x^2 + 1)^4\)  
c \(\frac{3x^2 + 1}{\sqrt{x^3 + x}}\)

**THINK**

a 1 Recognise that the derivative of \(x + 3\) is 1. Let \(u = x + 3\).

2 Find \(\frac{du}{dx}\).

3 Make \(dx\) the subject.

4 Substitute for \(x + 3\) and \(dx\).

5 Antidifferentiate with respect to \(u\).

6 Replace \(u\) with \(x + 3\) and state answer in terms of \(x\).

b 1 Recognise that \(4x\) is the derivative of \(2x^2 + 1\). Let \(u = 2x^2 + 1\).

2 Find \(\frac{du}{dx}\).

3 Make \(dx\) the subject.

4 Substitute \(u\) for \(2x^2 + 1\) and \(\frac{du}{4x}\) for \(dx\).

5 Simplify the integrand by cancelling out the \(4x\).

6 Antidifferentiate with respect to \(u\).

7 Replace \(u\) with \(2x^2 + 1\).

C 1 Recognise that \(3x^2 + 1\) is the derivative of \(x^3 + x\). Let \(u = x^3 + x\).

2 Find \(\frac{du}{dx}\).

**WRITE**

a Let \(u = x + 3\).

\[\frac{du}{dx} = 1\]

or \(dx = du\)

So \[\int (x + 3)^7 \, dx = \int u^7 \, du\]

\[= \frac{u^8}{8} + c\]

\[= \frac{(x + 3)^8}{8} + c\]

b Let \(u = 2x^2 + 1\).

\[\frac{du}{dx} = 4x\]

or \(\frac{dx}{4x} = du\)

So \[\int 4x(2x^2 + 1)^4 \, dx\]

\[= \int 4x \cdot u^4 \, du\]

\[= \frac{u^5}{5} + c\]

\[= \frac{(2x^2 + 1)^5}{5} + c\]

C Let \(u = x^3 + x\).

\[\frac{du}{dx} = 3x^2 + 1\]
**THINK**

3. Make \( dx \) the subject.

4. Substitute \( u \) for \( x^3 + x \) and \( \frac{du}{3x^2 + 1} \) for \( dx \).

5. Cancel out \( 3x^2 + 1 \).

6. Express the integrand in index form.

7. Antidifferentiate with respect to \( u \).

8. Replace \( u \) with \( x^3 + x \).

9. Express in root notation.

**WRITE**

or \( dx = \frac{du}{3x^2 + 1} \)

So \( \int \frac{3x^2 + 1}{\sqrt{x^3 + x}} \, dx \)

\( = \int \frac{3x^2 + 1}{\sqrt{u}} \times \frac{du}{3x^2 + 1} \)

\( = \int \frac{du}{\sqrt{u}} \)

\( = \int u^{\frac{1}{2}} \, du \)

\( = 2u^2 + c \)

\( = 2(x^3 + x)^{\frac{1}{2}} + c \)

\( = 2\sqrt{x^3 + x} + c \)

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**WORKED Example 2**

Antidifferentiate the following functions with respect to \( x \).

\( \text{a} \quad f(x) = \frac{x + 3}{(x^2 + 6x)^3} \quad \text{b} \quad f(x) = (x^2 - 1) \cos(3x - x^3) \)

**THINK**

\( \text{a} \quad 1 \) Express in integral notation.

2. Recognise that \( x + 3 \) is half of the derivative of \( x^2 + 6x \).

3. Let \( u = x^2 + 6x \).

4. Find \( \frac{du}{dx} \).

5. Make \( dx \) the subject.

6. Substitute \( u \) for \( x^2 + 6x \) and \( \frac{du}{2x + 6} \) for \( dx \).

7. Factorise \( 2x + 6 \).

8. Cancel out \( x + 3 \) and express \( u \) in index form on the numerator.

**WRITE**

\( \text{a} \quad \int \frac{x + 3}{(x^2 + 6x)^3} \, dx \)

Let \( u = x^2 + 6x \).

\( \frac{du}{dx} = 2x + 6 \)

or \( \frac{dx}{2x + 6} \)

So \( \int \frac{x + 3}{(x^2 + 6x)^3} \, dx = \int \frac{x + 3}{u^3} \times \frac{du}{2x + 6} \)

\( = \int \frac{x + 3}{u^3} \times \frac{du}{2(x + 3)} \)

\( = \int \frac{1}{2} u^{-3} \, du \)
THINK

9 Antidifferentiate with respect to \( u \).
10 Replace \( u \) with \( x^2 + 6x \).
11 Express the answer with a positive index number. (Optional.)

b 1 Express in integral notation.
2 Recognise that \( x^2 - 1 \) is a multiple of the derivative of \( 3x - x^3 \).
3 Let \( u = 3x - x^3 \).
4 Find \( \frac{du}{dx} \).
5 Make \( dx \) the subject.
6 Substitute \( u \) for \( 3x - x^3 \) and \( \frac{du}{3 - 3x^2} \) for \( dx \).
7 Factorise \( 3 - 3x^2 \).
8 Cancel out \( x^2 - 1 \).
9 Antidifferentiate with respect to \( u \).
10 Replace \( u \) with \( 3x - x^3 \).

WRITE

\[ \int (x^2 - 1) \cos(3x - x^3) \, dx \]

Let \( u = 3x - x^3 \).
\[ \frac{du}{dx} = 3 - 3x^2 \]
or \[ dx = \frac{du}{3 - 3x^2} \]
So
\[ \int (x^2 - 1) \cos(3x - x^3) \, dx = \int (x^2 - 1) \cos u \times \frac{du}{3 - 3x^2} = \int (x^2 - 1) \cos u \times \frac{du}{3(1 - x^2)} = \int (x^2 - 1) \cos u \times \frac{du}{-3(x^2 - 1)} = \frac{-\cos u}{3} \, du = -\sin u + c = -\sin \left(3x - x^3\right) + c \]

WORKED Example 3

Evaluate the following indefinite integrals.

\[ \int \cos x \sin^4 x \, dx \quad \int \frac{\tan^{-1}x}{2} \, dx \quad \int \frac{\log x}{x} \, dx \quad \int \sin^2 x \cos^3 x \, dx \]

THINK

a 1 Recognise that \( \cos x \) is the derivative of \( \sin x \).
2 Let \( u = \sin x \).

WRITE

\[ \int \cos x \sin^4 x \, dx \quad \int \frac{\tan^{-1}x}{2} \, dx \quad \int \frac{\log x}{x} \, dx \quad \int \sin^2 x \cos^3 x \, dx \]

Let \( u = \sin x \).
THINK

3. Find $\frac{du}{dx}$.

4. Make $dx$ the subject.

5. Substitute $u$ for $\sin x$ and $\frac{du}{\cos x}$ for $dx$.

6. Cancel out $\cos x$.

7. Antidifferentiate with respect to $u$.

8. Replace $u$ with $\sin x$.

b 1. Recognise that $\frac{1}{4 + x^2}$ is half of the derivative of $\tan^{-1} \frac{x}{2}$.

2. Let $u = \tan^{-1} \frac{x}{2}$.

3. Find $\frac{du}{dx}$.

4. Make $dx$ the subject.

5. Substitute $u$ for $\tan^{-1} \frac{x}{2}$ and $\frac{(4 + x^2)du}{2}$ for $dx$.

6. Cancel out $4 + x^2$.

7. Antidifferentiate with respect to $u$.

8. Replace $u$ with $\tan^{-1} \frac{x}{2}$.

b 1. Recognise that $\frac{1}{x}$ is half of the derivative of $\log_e 4x$.

2. Let $u = \log_e 4x$.

3. Find $\frac{du}{dx}$.

4. Make $dx$ the subject.

WRITE

\[ \frac{du}{dx} = \cos x \]

or $dx = \frac{du}{\cos x}$

So $\int \cos x \sin^4 x \, dx = \int (\cos x) u^4 \frac{du}{\cos x}$

$= \int u^4 \, du$

$= \frac{1}{5} u^5 + c$

$= \frac{1}{5} \sin^5 x + c$

b 1. Let $u = \tan^{-1} \frac{x}{2}$.

\[ \frac{du}{dx} = \frac{2}{4 + x^2} \]

or $dx = \frac{(4 + x^2)du}{2}$

So $\int \frac{\tan^{-1} \frac{x}{2}}{4 + x^2} \, dx$

$= \int \frac{u}{4 + x^2} \times \frac{(4 + x^2)du}{2}$

$= \int \frac{u}{2} \, du$

$= \frac{u^2}{4} + c$

$= \frac{(\tan^{-1} \frac{x}{2})^2}{4} + c$

c 1. Recognise that $\frac{1}{x}$ is the derivative of $\log_e 4x$.

2. Let $u = \log_e 4x$.

3. Find $\frac{du}{dx}$.

4. Make $dx$ the subject.

or $dx = x \, du$
THINK

5 Substitute $u$ for $\log_e 4x$ and $x\,du$ for $dx$ in the integral.

6 Cancel out $x$.

7 Antidifferentiate with respect to $u$.

8 Replace $u$ by $\log_e 4x$.

d1 Express $\cos^3 x$ as $\cos x \cos^2 x$.

2 Express $\cos x \cos^2 x$ as $\cos x (1 - \sin^2 x)$ (using the identity $\sin^2 x + \cos^2 x = 1$).

3 Let $u = \sin x$ as its derivative is a factor of the new form of the function.

4 Find $\frac{du}{dx}$.

5 Make $dx$ the subject.

6 Substitute $u$ for $\sin x$ and $\frac{du}{\cos x}$ for $dx$.

7 Cancel out $\cos x$.

8 Expand the integrand.

9 Antidifferentiate with respect to $u$.

10 Replace $u$ by $\sin x$.

WRITE

So $\int \frac{\log_e 4x}{x} \, dx$

$= \int \frac{u}{x} \times x \, du$

$= \int u \, du$

$= \frac{1}{2} u^2 + c$

$= \frac{1}{2} (\log_e 4x)^2 + c$

d $\int \sin^2 x \cos^3 x \, dx$

$= \int \sin^2 x \cos x \cos^2 x \, dx$

Let $u = \sin x$.

$\frac{du}{dx} = \cos x$

or $dx = \frac{du}{\cos x}$

So $\int \sin^2 x \cos^3 x \, dx$

$= \int u^2 \cos x (1 - u^2) \frac{du}{\cos x}$

$= \int u^2 (1 - u^2) \, du$

$= \int (u^2 - u^4) \, du$

$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + c$

$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + c$

WORKED Example 4

If $f'(x) = 4xe^{x^2}$ and $f(0) = 5$, find $f(x)$.

THINK

1 Express $f(x)$ in integral notation.

2 Recognise that $4x$ is twice the derivative of $x^2$.

3 Let $u = x^2$.

4 Find $\frac{du}{dx}$.

WRITE

$f(x) = \int 4xe^{x^2} \, dx$

Let $u = x^2$.

$\frac{du}{dx} = 2x$

Continued over page
Substitution where the derivative is present in the integrand

THINK

5 Make \( dx \) the subject.

6 Substitute \( u \) for \( x^2 \) and \( \frac{du}{2x} \) for \( dx \).

7 Cancel out \( 2x \).

8 Antidifferentiate with respect to \( u \).

9 Replace \( u \) by \( x^2 \).

10 Substitute \( x = 0 \) and \( f(0) = 5 \).

11 Solve for \( c \).

12 State the function \( f(x) \).

WRITE

or \( dx = \frac{du}{2x} \)

So \( f(x) = \int 4xe^u \frac{du}{2x} = \int 2e^u \, du \)

\( f(x) = 2e^u + c \)

\( f(0) = 2e^0 + c = 5 \)

\( 2 + c = 5 \)

\( c = 3 \)

Therefore \( f(x) = 2e^{x^2} + 3 \).

1. Since \( \frac{d[f(x)]^{n+1}}{dx} = (n + 1)f'(x)[f(x)]^n, n \neq -1 \)

then \( \int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n + 1} + c, n \neq -1 \).

2. Since \( \frac{d \log_e f(x)}{dx} = \frac{f'(x)}{f(x)} \)

then \( \int \frac{f'(x)}{f(x)} \, dx = \log_e f(x) + c \).

EXERCISE 6A

Substitution where the derivative is present in the integrand

1 Find the antiderivative for each of the following expressions.

- \( a \) \( 2x(x^2 + 3)^4 \)
- \( b \) \( 2x(6 - x^2)^{-3} \)
- \( c \) \( 3x^2(x^3 - 2)^5 \)
- \( d \) \( 2(x + 2)(x^2 + 4x)^{-3} \)
- \( e \) \( (2x + 5)\sqrt{x^2 + 5x} \)
- \( f \) \( \frac{2x - 3}{(x^2 - 3x)^4} \)
- \( g \) \( 3x^2(x^3 - 5)^2 \)
- \( h \) \( \frac{3x^2 + 4x}{\sqrt{x^3 + 2x^2}} \)
- \( i \) \( 4x^3e^{x^4} \)
- \( j \) \( (2x + 3)\sin(x^2 + 3x - 2) \)
- \( k \) \( (3x^2 + 5)\cos(x^3 + 5x) \)
- \( l \) \( \cos x \sin^3 x \)
Given that the derivative of \((x^2 + 5x)^4\) is \(4(x^2 + 5)(x^2 + 5x)^3\), then the antiderivative of 
\[ 8(x^2 + 5)(x^2 + 5x)^3 \]
is:

\[ A \quad 2(x^2 + 5x)^4 + c \quad B \quad \frac{1}{2} (x^2 + 5x)^4 + c \quad C \quad 4(x^2 + 5x)^4 + c \]

\[ D \quad 2(x^2 + 5x)^2 + c \quad E \quad \frac{1}{2} (x^2 + 5x)^2 + c \]

The integral \(\int \frac{x}{\sqrt{x^2 + 3}} \, dx\) can be found by making the substitution ‘\(u\)’ equal to:

\[ A \quad x^2 \quad B \quad x \quad C \quad \sqrt{x} \quad D \quad x^2 + 3 \quad E \quad 2x \]

After the appropriate substitution the integral becomes:

\[ A \quad \int u^{\frac{1}{2}} \, du \quad B \quad \frac{1}{2} \int u^{\frac{1}{2}} \, du \quad C \quad \frac{1}{2} \int (u + 3)^{\frac{3}{2}} \, du \]

\[ D \quad \int (u + 3)^{\frac{1}{2}} \, du \quad E \quad 2 \int u^{\frac{1}{2}} \, du \]

Hence the antiderivative of \(\frac{x}{x^2 + 3}\) is:

\[ A \quad \frac{2}{3} (x^2 + 3)^{\frac{3}{2}} + c \quad B \quad 4(x^2 + 3)^{\frac{1}{2}} + c \quad C \quad \frac{2}{3} (x^2 + 6)^{\frac{3}{2}} + c \]

\[ D \quad (x^2 + 6)^{\frac{1}{2}} + c \quad E \quad (x^2 + 3)^{\frac{1}{2}} + c \]

**Antidifferentiate each of the following expressions with respect to \(x\).**

\[ a \quad 6x^2(x^3 - 2)^5 \quad b \quad x(4 - x)^3 \]

\[ c \quad x^3(x^3 - 1)^7 \quad d \quad (x + 3)(x^2 + 6x - 2)^4 \]

\[ e \quad (x + 1)(x^2 + 2x + 3)^{-4} \quad f \quad \frac{4x + 6}{\sqrt{x^2 + 3x}} \]

\[ g \quad \frac{2x - 5}{(x^2 - 5x + 2)^6} \quad h \quad (x^2 - 1) \sqrt{4 - 3x + x^3} \]

\[ i \quad (6x - 3)e^{x^2 - x + 3} \quad j \quad x^2e^{x^2} \]

\[ k \quad (x + 1) \sin(x^2 + 2x - 3) \quad l \quad (x^2 - 2) \cos(6x - x^3) \]

\[ m \quad \sin 2x \cos 4x \quad n \quad \cos 3x \sin 3x \]

\[ o \quad \log_e 3x \quad p \quad \frac{(4x - 2) \log_e (x^2 - x)}{x^2 - x} \]

\[ q \quad \frac{\log_e x}{x} \quad r \quad \frac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}} \]

\[ s \quad -\sin^4 x \cos x \quad t \quad \sec^2 x \tan^3 x \]
Evaluate the following indefinite integrals.

5. a \( \int x(x^2 + 1)^{\frac{5}{2}} \, dx \)
   
   b \( \int x\sqrt{1 - x^2} \, dx \)

   c \( \int \frac{e^x(3 + 2e^x)^4}{4} \, dx \)

   d \( \int \frac{\sin x}{\cos^3 x} \, dx \)

   e \( \int x^2 \sin x^3 \, dx \)

   f \( \int \sin x \, e^{\cos x} \, dx \)

   g \( \int \frac{\cos x \log_e (\sin x)}{\sin x} \, dx \)

   h \( \int e^{3x}(1 - e^{3x})^2 \, dx \)

   i \( \int \frac{-2 \cos^{-1} \frac{x}{3}}{\sqrt{9 - x^2}} \, dx \)

   j \( \int (2x + 1) \sqrt{x^2 + 3} \, dx \)

   k \( \int (x + 1) \cos(x^2 + 4x) \, dx \)

   l \( \int \frac{e^{\sqrt{x + 1}}}{\sqrt{x + 1}} \, dx \)

   m \( \int \frac{\sin^{-1} 4x}{1 - 16x^2} \, dx \)

   n \( \int \frac{\tan^{-1} x}{1 + x^2} \, dx \)

   o \( \int \frac{x}{1 - 4x^2} \, dx \)

Find the antiderivative for each of the following expressions.

6. a \( \frac{\cos x}{\sqrt{1 + 3 \sin x}} \)

   b \( \sec^2 x \sqrt{2 + \tan x} \)

   c \( \sin x \sec^3 x \)

   d \( \frac{e^{2x}}{(e^{2x} - 3)^2} \)

   e \( \frac{\sec^2 x}{(5 - \tan x)^3} \)

   f \( \frac{4}{x \log_e x} \)

   g \( \frac{(\log_e x)^3}{x} \)

   h \( \frac{e^{\tan x}}{\cos^2 x} \)

   i \( \frac{e^x - e^{-x}}{\sqrt{e^x + e^{-x}}} \)

   j \( \frac{\sin x - \cos x}{\sin x + \cos x} \)

   k \( \frac{\sin^3 x \cos^2 x}{\sin x \cos x} \)

   l \( \frac{\log_e (\tan x)}{\sin x \cos x} \)

   m \( \frac{\log_e (\tan x)}{\sin x \cos x} \)

If \( f'(x) = \frac{x}{\sqrt{x^2 + 5}} \) and \( f(2) = 1 \) find \( f(x) \).

8. If \( f'(x) = \frac{e^{\sqrt{x}}}{\sqrt{x}} \) and \( f(0) = 3 \) find \( f(x) \).

9. If \( g(1) = -2 \) and \( g'(x) = \frac{4 \log_e x^2}{x} \) then find \( g(x) \).

10. If \( g\left(\frac{\pi}{4}\right) = 0 \) and \( g'(x) = 16 \sin x \cos^3 x \) then find \( g(x) \).
Technique 2: Linear substitution

For antiderivatives of the form \( \int f(x)[g(x)]^n \, dx \), where \( g(x) \) is a linear function, that is, of the type \( mx + c \), and \( f(x) \) is not the derivative of \( g(x) \), the substitution \( u = g(x) \) is often successful in finding the integral. Examples of this type of integral are:

1. \( \int \sqrt{4x + 1} \, dx \). In this example \( f(x) = 1 \) and \( g(x) = 4x + 1 \) with \( n = \frac{1}{2} \). By letting \( u = 4x + 1 \), and consequently \( dx = \frac{1}{4} \, du \), the integral becomes \( \frac{1}{4} \int \sqrt{u} \, du \) which can be readily antiderivated.

2. \( \int 4x(x - 3)^4 \, dx \). In this example \( f(x) = 4x \) and \( g(x) = x - 3 \) with \( n = 4 \). By letting \( u = x - 3 \), the function \( f(x) \) can be written in terms of \( u \), that is, \( u = x - 3 \), thus \( 4x = 4(u + 3) \) and further, \( dx = du \). The integral becomes \( \int 4(u + 3) \times u \, du \) which can be readily antiderivated.

The worked examples below illustrate how the use of the substitution \( u = g(x) \) simplifies integrals of the type \( \int f(x)[g(x)]^n \, dx \).

WORKED Example 5

i Using the appropriate substitution, express the following integrals in terms of \( u \) only.

ii Evaluate the integrals as functions of \( x \).

\[ \text{a} \quad \int x(x - 2)^{\frac{5}{2}} \, dx \quad \text{b} \quad \int \frac{x^2}{\sqrt{x} + 1} \, dx \]

**THINK**

\[ \text{a} \quad \text{i} \quad \begin{align*} &1 \quad \text{Let } u = x - 2. \\ &2 \quad \text{Find } \frac{du}{dx}. \\ &3 \quad \text{Make } dx \text{ the subject.} \\ &4 \quad \text{Substitute } u \text{ for } x - 2, u + 2 \text{ for } x \text{ and } du \text{ for } dx. \\ &5 \quad \text{Expand the integrand.} \\ &\text{ii} \quad \text{Antidifferentiate with respect to } u. \\ &2 \quad \text{Replace } u \text{ with } x - 2. \end{align*} \]

**WRITE**

\[ \text{a} \quad \text{i} \quad \begin{align*} &\text{Let } u = x - 2 \text{ and } x = u + 2. \\ &\frac{du}{dx} = 1 \\ &dx = du \\ &\text{So } \int x(x - 2)^{\frac{5}{2}} \, dx \\ &= \int (u + 2)^{\frac{5}{2}} \, du \\ &= \int \left( u^\frac{7}{2} + 2u^{\frac{5}{2}} \right) \, dx \\ &= \frac{9}{3}u^\frac{9}{2} + \frac{4}{7}u^\frac{7}{2} + c \\ &= \frac{3}{5}(x - 2)^\frac{9}{2} + \frac{4}{7}(x - 2)^\frac{7}{2} + c \\ &\text{Continued over page} \end{align*} \]
THINK

3 Take out the factor of 
\( \frac{7}{2(x - 2)^3} \).

4 Simplify the other factor.

WRITE

\[ = 2(x - 2)^\frac{7}{3} \left( \frac{x - 2}{9} + \frac{4}{7} \right) + c \]
\[ = 2(x - 2)^\frac{7}{3} \left( \frac{7x - 14 + 36}{63} \right) + c \]
\[ = 2(x - 2)^\frac{7}{3} \left( \frac{7x + 22}{63} \right) + c \]

b i 1 Express \( x + 1 \) in index form.

2 Let \( u = x + 1 \).

3 Find \( \frac{du}{dx} \).

4 Make \( dx \) the subject.

5 Express \( x \) in terms of \( u \).

6 Hence express \( x^2 \) in terms of \( u \).

7 Substitute \( u \) for \( x + 1 \), \( u^2 - 2u + 1 \) for \( x^2 \) and \( du \) for \( dx \).

8 Expand the integrand.

b i \[ \int \frac{x^2}{\sqrt{x + 1}} \, dx \]

Let \( u = x + 1 \).

or \( dx = du \)

\[ = \int (u^2 - 2u + 1)^{-\frac{1}{2}} \, du \]

\[ = \int \left( \frac{1}{u^2} - 2u^2 + u \right)^{\frac{1}{2}} \, du \]

\[ \text{So} \int x^2 (x + 1)^{-\frac{1}{2}} \, dx \]

\[ = \int (u^2 - 2u + 1) u^{-\frac{1}{2}} \, du \]

\[ = \int \left( \frac{u^{\frac{3}{2}}}{2} - \frac{u^{\frac{5}{2}}}{3} + 2u^{\frac{1}{2}} \right) du \]

ii 1 Antidifferentiate with respect to \( u \).

2 Replace \( u \) with \( x + 1 \).

3 Take \( 2(x + 1)^{\frac{1}{2}} \) out as a factor.

4 Simplify the other factor.

\[ = 2(x + 1)^{\frac{1}{2}} \left[ \frac{(x + 1)^2}{5} - \frac{2(x + 1)}{3} + 1 \right] + c \]
\[ = 2(x + 1)^{\frac{1}{2}} \left[ \frac{(x^2 + 2x + 1)}{5} + \frac{(-2x - 2)}{3} + 1 \right] + c \]
\[ = 2(x + 1)^{\frac{1}{2}} \left[ \frac{3x^2 + 6x + 3 - 10x - 10 + 15}{15} \right] + c \]
\[ = 2(x + 1)^{\frac{1}{2}} \left[ \frac{3x^2 - 4x + 8}{15} \right] + c \]
**WORKED Example 6**

**a** Find the antiderivative of \( \frac{e^{2x}}{e^x + 1} \).

**WRITE**

**a** Let \( u = e^x + 1 \).

\[
\frac{du}{dx} = e^x \quad \text{or} \quad dx = \frac{du}{e^x}
\]

and \( e^x = u - 1 \)

So

\[
\int \frac{e^{2x}}{e^x + 1} \, dx = \int \frac{e^x}{u} \times \frac{du}{e^x} = \int \frac{e^x}{u} \, du = \int \left( \frac{u - 1}{u} \right) \, du = u - \log_e u + c = e^x + 1 - \log_e (e^x + 1) + c
\]

**b** State the domain of the antiderivative.

**b** For \( \log_e (e^x + 1) \) to exist \( e^x + 1 > 0 \), which is for all \( x \).

Therefore the domain of the integral is \( \mathbb{R} \).

**Note:** Recall that the logarithm of a negative number cannot be found.

---

**remember**

For antiderivatives of the form \( \int f(x)[g(x)]^n \, dx, n \neq 0 \), make the substitution \( u = g(x) \) and so \( [g(x)]^n \, dx, n \neq 0 \) becomes \( g'(x)u^n \, du, n \neq 0 \). This technique can be used for the specific case where \( g = mx + c \) since \( g'(x) = m \). The function \( f(x) \) needs to be transformed in terms of the variable \( u \) as well.
**EXERCISE 6B**  

**Linear substitution**

1. By making the appropriate substitution for \( u \):
   - i express the following integrals in terms of \( u \)
   - ii evaluate the integrals as functions of \( x \).

   a \[ \int \frac{4}{x - 3} \, dx \]
   b \[ \int \frac{2}{3x + 5} \, dx \]
   c \[ \int \sqrt[4]{x + 1} \, dx \]
   d \[ \int \sqrt[3]{3 - 2x} \, dx \]
   e \[ \int x(x + 1)^3 \, dx \]
   f \[ \int 4x(x - 3)^4 \, dx \]
   g \[ \int 2x(2x + 1)^4 \, dx \]
   h \[ \int 3x(1 - 3x)^5 \, dx \]
   i \[ \int 6x(3x - 2)^3 \, dx \]
   j \[ \int x(2x + 7)^{\frac{1}{3}} \, dx \]
   k \[ \int x\sqrt{x + 3} \, dx \]
   l \[ \int x\sqrt{3x - 4} \, dx \]
   m \[ \int (x + 2)(x - 4)^3 \, dx \]
   n \[ \int (x - 3)(2x + 1)^{\frac{5}{2}} \, dx \]
   o \[ \int \frac{2x}{\sqrt{x - 6}} \, dx \]
   p \[ \int \frac{3x}{\sqrt{8 - x}} \, dx \]

2. **multiple choice**
   - a The integral \( \int 4x\sqrt{x + 2} \, dx \) can be found by letting \( u \) equal:
     - A \( \sqrt{x + 2} \)
     - B \( x \)
     - C \( x + 2 \)
     - D \( 4x \)
     - E \( 2x \)
   - b The integral then becomes:
     - A \( \int u^5 \, du \)
     - B \( \int \left(2u^{\frac{1}{2}} - 4u^{\frac{1}{3}}\right) \, du \)
     - C \( \int 2u^{\frac{1}{2}} \, du \)
     - D \( \int \left(4u^{\frac{1}{2}} - 2u^{\frac{3}{2}}\right) \, du \)
     - E \( \int \left(4u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right) \, du \)

3. **multiple choice**
   - a Using the appropriate substitution, \( \int \frac{x^2}{\sqrt{x - 1}} \, dx \) becomes:
     - A \( \int u \, du \)
     - B \( \int \left(u^{\frac{1}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) \, du \)
     - C \( \int \left(u^{\frac{1}{2}} + 2u^{\frac{3}{2}}\right) \, du \)
     - D \( \int \left(u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) \, du \)
     - E \( \int u^3 \, du \)
b The result of the integration is:

\[ A \frac{2}{3}(x - 1)^{\frac{3}{2}} + c \]

\[ B \frac{2}{3}(x - 1)^{\frac{3}{2}} + 4(x - 1)^{\frac{1}{2}} + c \]

\[ C \frac{2}{5}(x - 1)^{\frac{5}{2}} + c \]

\[ D \frac{2}{3}(x - 1)^{\frac{5}{2}} + \frac{4}{3}(x - 1)^{\frac{3}{2}} + 2(x - 1)^{\frac{1}{2}} + c \]

\[ E \frac{2}{3}(x - 1)^{\frac{7}{2}} + \frac{4}{5}(x - 1)^{\frac{5}{2}} + \frac{4}{3}(x - 1)^{\frac{3}{2}} + c \]

Worked Example 6a

Find the antiderivative of each of the following expressions.

\[ a \ x^2(x - 4)^4 \]
\[ b \ x^2(5 - x)^3 \]
\[ c \ x^2 \sqrt{x - 1} \]
\[ d \ x^2 \sqrt{3 - x} \]
\[ e \ x^2(x + 2)^{\frac{3}{2}} \]
\[ f \ x^2(1 - x)^{\frac{3}{2}} \]
\[ g \ (x + 1)^2 \sqrt{x - 2} \]
\[ h \ (x - 3)^2 \sqrt{x + 1} \]
\[ i \ \frac{e^x}{e^x + 1} \]
\[ j \ \frac{x^2}{\sqrt{x + 1}} \]
\[ k \ \frac{2x^2}{\sqrt{3 - x}} \]
\[ l \ x^3 \sqrt{x - 1} \]
\[ m \ \frac{x^3}{\sqrt{x + 4}} \]
\[ n \ \frac{2x^3}{\sqrt{1 - x}} \]
\[ o \ \frac{x + 3}{(x - 2)^2} \]
\[ p \ \frac{2x - 1}{(x + 1)^3} \]
\[ q \ \frac{4x}{(x + 2)^2} \]
\[ r \ \frac{x^2}{(x - 1)^2} \]
\[ s \ \frac{(x + 3)^2}{\sqrt{x + 2}} \]
\[ t \ \frac{(x - 2)^2}{\sqrt{2 - x}} \]
\[ u \ \frac{e^{2x}}{e^x + 2} \]
\[ v \ \frac{e^{3x}}{e^x - 1} \]

Worked Example 6b

a If \( f'(x) = -(5 - x)^{\frac{1}{2}} + 10(5 - x)^{-\frac{1}{2}} \) and \( f(1) = -2 \), find \( f(x) \).

b State the domain of \( f(x) \).

6a If \( f'(x) = \frac{5(x + 1)^{\frac{3}{2}}}{2} - 3(x + 1)^{\frac{1}{2}} + \frac{(x + 1)^{-\frac{1}{2}}}{2} \) and \( f(0) = 1 \), find \( f(x) \).

b State the domain of \( f(x) \).

7a Given that \( g'(x) = \frac{2x + 1}{(x - 1)^2} \) and \( g(2) = 0 \), find \( g(x) \).

b State the domain of \( g(x) \).

8a Given that \( g(0) = 2 - \log_2 2 \) and \( g'(x) = \frac{e^{2x}}{e^x + 1} \), find \( g(x) \).

b State the domain of \( g(x) \).
Technique 3: Antiderivatives involving trigonometric identities

Different trigonometric identities can be used to antidifferentiate \( \sin^n x \) and \( \cos^n x \); \( n \in J^+ \) depending on whether \( n \) is even or odd. Functions involving \( \tan^2 ax \) are also discussed.

**Even powers of \( \sin x \) or \( \cos x \)**

The double-angle trigonometric identities can be used to antidifferentiate even powers of \( \sin x \) or \( \cos x \). The first identity is:

\[
\cos 2x = 1 - 2 \sin^2 x \\
= 2 \cos^2 x - 1
\]

Therefore

\[
\sin^2 x = \frac{1}{2} (1 - \cos 2x)
\]

or

\[
\cos^2 x = \frac{1}{2} (1 + \cos 2x)
\]

The second identity is:

\[
\sin 2x = 2 \sin x \cos x
\]

or

\[
\sin x \cos x = \frac{1}{2} \sin 2x
\]

These may be expressed in the following general forms:

\[
\begin{align*}
\sin^2 ax &= \frac{1}{2} (1 - \cos 2ax) \\
\cos^2 ax &= \frac{1}{2} (1 + \cos 2ax) \\
\sin ax \cos ax &= \frac{1}{2} \sin 2ax
\end{align*}
\]

**WORKED Example 7**

Find the antiderivative of the following expressions.

a \( \sin^2 \frac{x}{2} \)  

b \( 2 \cos^2 \frac{x}{4} \)

**THINK**

a 1. Express in integral notation.

2. Use identity 1 to change \( \sin^2 \frac{x}{2} \).

3. Take the factor of \( \frac{1}{2} \) to the front of the integral.

4. Antidifferentiate by rule.

5. Simplify the answer.

b 1. Express in integral notation.

**WRITE**

a \[ \int \sin^2 \frac{x}{2} \, dx \]

\[ = \int \frac{1}{2} (1 - \cos x) \, dx \]

\[ = \frac{1}{2} \left( x - \sin x \right) + c \]

b \[ \int 2 \cos^2 \frac{x}{4} \, dx \]
Odd powers of \( \sin x \) or \( \cos x \)

For integrals involving odd powers of \( \sin x \) or \( \cos x \) the identity:

\[
\sin^2 x + \cos^2 x = 1
\]

can be used so that the ‘derivative method’ of substitution then becomes applicable. The following worked example illustrates the use of this identity whenever there is an odd-powered trigonometric function in the integrand.
Find the antiderivative of the following expressions.

\( a \cos^3 x \quad b \cos x \sin 2x \quad c \cos^4 2x \sin^3 2x \)

**THINK**

1. Express in integral notation.
2. Factorise \( \cos^3 x \) as \( \cos x \cos^2 x \).
3. Use the identity: \( 1 - \sin^2 x \) for \( \cos^2 x \).
4. Let \( u = \sin x \) so the derivative method can be applied.
5. Find \( \frac{du}{dx} \).
6. Make \( dx \) the subject.
7. Substitute \( u \) for \( \sin x \) and \( \frac{du}{\cos x} \) for \( dx \).
8. Cancel out \( \cos x \).
9. Antidifferentiate with respect to \( u \).
10. Replace \( u \) with \( \sin x \).

**WRITE**

\[ a \int \cos^3 x \, dx \]
\[ = \int \cos x \cos^2 x \, dx \]
\[ = \int \cos x (1 - \sin^2 x) \, dx \]
Let \( u = \sin x \).
\[ = \int (1 - u^2) \, du \]
\[ = u - \frac{1}{3} u^3 + c \]
\[ = \sin x - \frac{1}{3} \sin^3 x + c \]

\[ b \int \cos x \sin 2x \, dx \]
\[ = \int \cos x (2 \sin x \cos x) \, dx \]
\[ = \int 2 \sin x \cos^2 x \, dx \]
Let \( u = \cos x \).
\[ \frac{du}{dx} = -\sin x \]
\[ = \int 2 \sin x (u^2) \frac{du}{-\sin x} \]
So \[ 2 \sin x \cos^2 x \, dx \]

\[ = 2 \sin x \left( u^2 \right) \frac{du}{-\sin x} \]
Using the identity \( \sec^2 x = 1 + \tan^2 x \)

The identity \( \sec^2 ax = 1 + \tan^2 ax \) is used to antidifferentiate expressions involving \( \tan^2 ax + c \) where \( c \) is a constant since the antiderivative of \( \sec^2 x \) is \( \tan x \).

Otherwise, expressions of the form \( \tan^n x \sec^2 x \) can be antidifferentiated using the ‘derivative method’ of exercise 6A.
Find an antiderivative for each of the following expressions.

\[ a \int (2 + \tan^2 x) \, dx \quad b \int 3 \tan^2 3x \sec^2 3x \, dx \]

**THINK**

1. Express \(2 + \tan^2 x\) as \(1 + \sec^2 x\) using the identity.

2. Antidifferentiate by rule. There is no need to add \(c\) as one antiderivative only is required.

**WRITE**

\[ a \int (2 + \tan^2 x) \, dx \]

\[ = \int (1 + \sec^2 x) \, dx \]

\[ = x + \tan x \]

\[ b \int 3 \tan^2 3x \sec^2 3x \, dx \]

Let \(u = \tan 3x\) so that the derivative method can be applied.

1. Let \(u = \tan 3x\).

2. Find \(\frac{du}{dx}\).

3. Make \(dx\) the subject.

4. Substitute \(u\) for \(\tan 3x\) and \(\frac{du}{3 \sec^2 3x}\) for \(dx\).

5. Cancel out \(3\sec^2 3x\).

6. Antidifferentiate with respect to \(u\).

7. Replace \(u\) with \(\tan 3x\).

\[ \int 3 \tan^2 3x \sec^2 3x \, dx \]

\[ = \int 3 u^2 \sec^2 3x \frac{du}{3\sec^2 3x} \]

\[ = \int u^2 \, du \]

\[ = \frac{1}{3} u^3 \]

\[ = \frac{1}{3} \tan^3 3x \]

**remember**

1. Trigonometric identities can be used to antidifferentiate odd and even powers of \(\sin x\) and \(\cos x\). These identities are:

   \[
   \sin^2 ax = \frac{1}{2} (1 - \cos 2ax) \\
   \cos^2 ax = \frac{1}{2} (1 + \cos 2ax) \\
   \sin ax \cos ax = \frac{1}{2} \sin 2ax
   \]

2. The identity \(\sec^2 ax = 1 + \tan^2 ax\) is used to antidifferentiate expressions involving \(\tan^2 ax + c\) where \(c\) is a constant.
Antiderivatives involving trigonometric identities

1. Antidifferentiate each of the following expressions with respect to $x$.
   - $a \cos^2 x$
   - $b \sin^2 x$
   - $c 2 \cos^4 x$
   - $d 4 \sin^2 3x$
   - $e \cos^2 5x$
   - $f \sin^2 6x$
   - $g \cos^2 x \over 2$
   - $h \sin^2 x \over 3$
   - $i 3 \cos^2 x \over 6$
   - $j 2 \sin^2 x \over 4$
   - $k \cos^2 2x \over 3$
   - $l \sin^2 3x \over 2$

2. Evaluate the following indefinite integrals as functions of $x$.
   - $a \int 2 \sin x \cos x \, dx$
   - $b \int 4 \sin 2x \cos 2x \, dx$
   - $c \int \sin 3x \cos 3x \, dx$
   - $d \int -2 \sin 4x \cos 4x \, dx$
   - $e \int \sin^2 x \cos^2 x \, dx$
   - $f \int \sin^2 2x \cos^2 2x \, dx$
   - $g \int 2 \sin^2 4x \cos^2 4x \, dx$
   - $h \int 2 \sin^2 3x \cos^2 3x \, dx$
   - $i \int 6 \sin^2 x \over 2 \cos^2 x \over 2 \, dx$
   - $j \int 4 \sin^2 x \over 3 \cos^2 x \over 3 \, dx$
   - $k \int \sin^2 5x \over 2 \cos^2 5x \over 2 \, dx$
   - $l \int -2 \sin^2 4x \over 3 \cos^2 4x \over 3 \, dx$

3. Multiple choice
   If $a$ is a constant, then,
   - $a \int \sin^2 ax \, dx$ is equal to:
     - $A \ 2x - \sin 2ax + c$
     - $B \ x - 2a \sin 2ax + c$
     - $C \ {x \over 2} - {\sin 2ax \over 4a} + c$
     - $D \ x - {1 \over 2} \sin ax \over 2 + c$
     - $E \ {x \over 2} - {1 \over a} \sin ax \over 2 + c$
   - $b \int \sin^2 ax \cos^2 ax \, dx$ is equal to:
     - $A \ {x \over 8} - {\sin 4ax \over 32a} + c$
     - $B \ {x \over 2} - {\sin ax \over 4a} + c$
     - $C \ {x \over 4} - {\cos ax \over 8a} + c$
     - $D \ x - {\sin 4ax \over 16a} + c$
     - $E \ {x \over 8} + {\cos 4ax \over 16a} + c$
   - $c \int \cos^3 ax \, dx$ is equal to:
     - $A \ a \cos ax - 3a \cos^3 ax + c$
     - $B \ a \sin ax - 3 \cos^3 ax + c$
     - $C \ {\cos^4 ax \over 4a} + c$
     - $D \ {\sin^4 ax \over 4a} + c$
     - $E \ {1 \over 3a} (3 \sin ax - \sin^3 ax) + c$
4 Find an antiderivative of each of the following expressions.

\[ \begin{align*}
\text{a) } & \sin^3 x & \text{b) } & \cos^2 x & \text{c) } & 6 \sin^4 x & \text{d) } & 4 \cos^3 x \\
\text{e) } & \sin^3 7x & \text{f) } & \cos^3 6x & \text{g) } & 3 \sin^3 \frac{x}{2} & \text{h) } & 2 \cos^3 \frac{x}{3} \\
\text{i) } & \sin^3 \frac{3x}{2} & \text{j) } & \cos^3 \frac{5x}{2} & \text{k) } & \sin^3 \frac{3x}{4} & \text{l) } & \cos^3 \frac{4x}{3}
\end{align*} \]

5 Use the appropriate identities to antidifferentiate the following expressions.

\[ \begin{align*}
\text{a) } & \sin x \cos 2x & \text{b) } & \cos 2x \cos 4x & \text{c) } & \sin 3x \cos 6x \\
\text{d) } & \cos 4x \cos 8x & \text{e) } & \sin^\frac{x}{2} \cos x & \text{f) } & \cos^\frac{x}{3} \cos^\frac{2x}{3}
\end{align*} \]

6 Antidifferentiate each of the following expressions with respect to \( x \).

\[ \begin{align*}
\text{a) } & \sin x \cos^4 x & \text{b) } & \sin 2x \cos^2 x & \text{c) } & \sin^\frac{x}{2} \cos^\frac{5x}{2} \\
\text{d) } & \cos 3x \sin^4 3x & \text{e) } & \cos^\frac{x}{3} \sin^\frac{5x}{3} & \text{f) } & \cos^\frac{2x}{3} \sin^\frac{7x}{3}
\end{align*} \]

7 Find the following integrals.

\[ \begin{align*}
\text{a) } & \int \cos^2 x \sin^3 x \, dx & \text{b) } & \int \sin^2 x \cos^3 x \, dx \\
\text{c) } & \int \cos^2 x \sin^2 x \, dx & \text{d) } & \int \sin^2 x \cos^3 3x \, dx \\
\text{e) } & \int \cos^2 x \sin^3 \frac{x}{2} \, dx & \text{f) } & \int \sin^2 \frac{3x}{2} \cos^3 \frac{3x}{2} \, dx \\
\text{g) } & \int 4 \cos^\frac{x}{3} \sin^\frac{5x}{3} \, dx & \text{h) } & \int -6 \sin^2 \frac{5x}{4} \cos^3 \frac{5x}{4} \, dx \\
\text{i) } & \int \sin^3 x \cos^4 \, dx & \text{j) } & \int \cos^3 2x \sin^4 2x \, dx \\
\text{k) } & \int 2 \sin^3 2x \cos^5 2x \, dx & \text{l) } & \int -2 \cos^3 3x \sin^6 3x \, dx \\
\text{m) } & \int 4 \sin^\frac{3x}{2} \cos^6 \frac{x}{2} \, dx & \text{n) } & \int \cos^3 \frac{3x}{2} \sin^7 \frac{3x}{2} \, dx
\end{align*} \]

8 Find an antiderivative for each of the following expressions:

\[ \begin{align*}
\text{a) } & 1 + \tan^2 2x & \text{b) } & 1 + \tan^2 \frac{x}{3} & \text{c) } & \tan^2 x \sec^2 x \\
\text{d) } & \tan^3 x \sec^2 x & \text{e) } & 4 \tan^5 2x \sec^2 2x & \text{f) } & 8 \tan^\frac{4x}{2} \sec^2 \frac{x}{2} \\
\text{g) } & \tan^2 x \sec^4 x & \text{h) } & 6 \tan^2 x \sec^4 2x & \text{i) } & 2 \tan^\frac{2x}{2} \sec^4 \frac{x}{2} \\
\text{j) } & 3 \tan^3 x \sec^4 3x & \text{k) } & \tan^\frac{4x}{5} \sec^4 \frac{x}{5} & \text{l) } & 12 \tan^6 x \sec^6 x
\end{align*} \]

9 Find the following integrals where \( n \in \mathbb{J}^+ \).

\[ \begin{align*}
\text{a) } & \int \sin x \cos^n x \, dx & \text{b) } & \int \cos x \sin^n x \, dx & \text{c) } & \int \sec^2 x \tan^n x \, dx \\
\text{d) } & \int \sin^3 x \cos^n x \, dx & \text{e) } & \int \cos^3 x \sin^n x \, dx
\end{align*} \]
10 If \( f'(x) = 6 \sin x \cos^2 x \) and \( f\left(\frac{\pi}{3}\right) = 0 \), find \( f(x) \).

11 If \( f'(x) = 4 \sin^2 2x \cos^2 2x \) and \( f\left(\frac{\pi}{4}\right) = \pi \), find \( f(x) \).

12 Find \( g(x) \) if \( g'(x) = \sin^3 \frac{x}{2} \cos^4 \frac{x}{2} \) and \( g(0) = -\frac{4}{35} \).

The graph of a function and the graphs of its antiderivatives

Given \( f(x) = 2\cos^2 \frac{x}{2} \), what do the graphs of its antiderivatives look like?

Using a graphics calculator, press \( \text{Y=} \), enter \( Y1= 2(\cos(X/2))^2 \), move down to \( Y2= \), press \( \text{MATH} \) and select \( 9:\text{fnInt} \). Complete to obtain \( \text{fnInt}(Y1,X,0,X) \).

(Remember that to insert the symbol \( Y1 \), press \( \text{VARS} \), select \( \text{Y-VARS} \) and \( 1:\text{Function} \). Then select \( 1:Y1 \) and press \( \text{ENTER} \) (and similarly for any \( Y \) variable).

As the given function is trigonometric, press \( \text{ZOOM} \) and select \( 7: \text{ZTrig} \).

(Since the numeric integral is repeatedly applied for every \( X \)-value on the screen, the antiderivative graph can take some time to plot. You can speed it up considerably by changing the value of \( Xres \) in the \( \text{WINDOW} \) settings to 5.)

1 Which is the graph of \( f(x) = 2\cos^2 \frac{x}{2} \) and which is the graph of the antiderivative?

The antiderivative graph in the second screen is the line that cuts 0 at \( x = 0 \), since the integral from 0 to 0 of any function is 0. To see another antiderivative graph, go to \( Y3= \), press \( \text{MATH} \), select \( 9 \) and complete \( 9:\text{fnInt}(Y1,X,1,X) \) and then press \( \text{GRAPH} \).

2 Generate another two antiderivative graphs on your calculator. Sketch the function and the four antiderivative graphs. Describe any relationships you can find.

3 Choose another function and investigate the relationship between the graph of the function and the graphs of its antiderivatives.
Technique 4: Antidifferentiation using partial fractions

Recall that rational expressions, in particular those with denominators that can be expressed with linear factors, can be transformed into partial fractions. A summary of two common transformations is shown in the table below. These transformations are useful when the degree of the numerator is less than the degree of the denominator; otherwise long division is generally required before antidifferentiation can be performed.

<table>
<thead>
<tr>
<th>Rational expression</th>
<th>Equivalent partial fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{f(x)}{(ax + b)(cx + d)} ) where ( f(x) ) is a linear function</td>
<td>( \frac{A}{(ax + b)} + \frac{B}{(cx + d)} )</td>
</tr>
<tr>
<td>( \frac{f(x)}{(ax + b)^2} ) where ( f(x) ) is a linear function</td>
<td>( \frac{A}{(ax + b)^2} + \frac{B}{(ax + b)} )</td>
</tr>
</tbody>
</table>

We have seen how this procedure simplifies the sketching of graphs of rational functions. Similarly, expressing rational functions as partial fractions enables them to be antidifferentiated quite easily. However, it is preferable to use a substitution method, if it is applicable, as the partial-fraction technique can be tedious.

**WORKED Example 11**

Find \( a, b \) and \( c \) if \( ax(x - 2) + bx(x + 1) + c(x + 1)(x - 2) = 2x - 4 \).

**THINK**

1. Let \( x = 0 \) so that \( c \) can be evaluated. Let \( x = 0, \ -2c = -4 \)
2. Solve the equation for \( c \). \( c = 2 \)
3. Let \( x = 2 \) so that \( b \) can be evaluated. Let \( x = 2, \ 6b = 0 \)
4. Solve the equation for \( b \). \( b = 0 \)
5. Let \( x = -1 \) so that \( a \) can be evaluated. Let \( x = -1, \ 3a = -6 \)
6. Solve the equation for \( a \). \( a = -2 \)
7. State the solution. Therefore \( a = -2, \ b = 0 \) and \( c = 2 \).

**WORKED Example 12**

For each of the following rational expressions:

i express as partial fractions

ii antidifferentiate the result.

a \( \frac{x + 7}{(x + 2)(x - 3)} \)  

b \( \frac{2x - 3}{x^2 - 3x - 4} \)

**THINK**

a i 1. Express the rational expression as two separate fractions with denominators \( (x + 2) \) and \( (x - 3) \) respectively.

2. Express the partial fractions with the original common denominator.

**WRITE**

a i \( \frac{x + 7}{(x + 2)(x - 3)} = \frac{a}{(x + 2)} + \frac{b}{(x - 3)} \)  

\[ = \frac{a(x - 3) + b(x + 2)}{(x + 2)(x - 3)} \]
THINK

3 Equate the numerator on the left-hand side with the right-hand side.
4 Let \( x = -2 \) so that \( a \) can be evaluated.
5 Solve for \( a \).
6 Let \( x = 3 \) so that \( b \) can be evaluated.
7 Solve for \( b \).
8 Rewrite the rational expression as partial fractions.

WRITE

so \( x + 7 = a(x - 3) + b(x + 2) \)

Let \( x = -2 \), and thus \( 5 = -5a \)

\( a = 1 \)

Let \( x = 3 \), and thus \( 10 = 5b \)

\( b = 2 \)

Therefore \( \frac{x + 7}{(x + 2)(x - 3)} = \frac{-1}{x + 2} + \frac{2}{x - 3} \)

ii \( \int \frac{x + 7}{(x + 2)(x - 3)} \, dx \)

= \( \int \left( \frac{-1}{x + 2} + \frac{2}{x - 3} \right) \, dx \)

= \( -\log_e(x + 2) + 2 \log_e(x - 3) + c \) \((x > 3)\)

= \( \log_e \left( \frac{(x - 3)^2}{x + 2} \right) + c \)

b i 1 Factorise the denominator.

2 Express the partial fractions with denominators \((x - 4)\) and \((x + 1)\) respectively.

3 Express the right-hand side with the original common denominator.

4 Equate the numerators.

5 Let \( x = 4 \) to evaluate \( a \).

6 Solve for \( a \).

7 Let \( x = -1 \) to evaluate \( b \).

8 Solve for \( b \).

9 Rewrite the rational expression as partial fractions.

ii 1 Express the integral in its partial fraction form.

2 Antidifferentiate by rule.

3 Simplify using log laws.

b i \( \frac{2x - 3}{x^2 - 3x - 4} = \frac{2x - 3}{(x - 4)(x + 1)} \)

= \( \frac{a}{x - 4} + \frac{b}{x + 1} \)

= \( \frac{a(x + 1) + b(x - 4)}{(x - 4)(x + 1)} \)

So \( 2x - 3 = a(x + 1) + b(x - 4) \)

Let \( x = 4 \), \( 5 = 5a \)

\( a = 1 \)

Let \( x = -1 \), \( -5 = -5b \)

\( b = 1 \)

Therefore \( \frac{2x - 3}{x^2 - 3x - 4} = \frac{1}{x - 4} + \frac{1}{x + 1} \)

ii \( \int \frac{2x - 3}{x^2 - 3x - 4} \, dx \)

= \( \int \left( \frac{1}{x - 4} + \frac{1}{x + 1} \right) \, dx \)

= \( \log_e(x - 4) + \log_e(x + 1) + c \) \((x > 4)\)

= \( \log_e((x - 4)(x + 1)) + c \) \((x > 4)\)

or \( \log_e(x^2 - 3x - 4) + c \) \((x > 4)\)
Find the following integrals.

\[ \int \frac{2}{1 - x^2} \, dx \quad \int \frac{x^2 + 6x - 1}{(x + 4)(x + 1)} \, dx \]

**THINK**

1. Factorise the denominator of the integrand.

2. Express into partial fractions with denominators \((1 - x)\) and \((1 + x)\).

3. Express the partial fractions with the original common denominator.

4. Equate the numerators.

5. Let \(x = 1\) to find \(a\).

6. Solve for \(a\).

7. Let \(x = -1\) to find \(b\).

8. Solve for \(b\).

9. Express the integrand in its partial fraction form.

10. Antidifferentiate by rule.


**WRITE**

\[ \frac{2}{1 - x^2} = \frac{2}{(1 - x)(1 + x)} = \frac{a}{1 - x} + \frac{b}{1 + x} = \frac{a(1 + x) + b(1 - x)}{1 - x^2} \]

so \(2 = a(1 + x) + b(1 - x)\)

Let \(x = 1\), \(2 = 2a\)

\[ a = 1 \]

Let \(x = -1\), \(2 = 2b\)

\[ b = 1 \]

Therefore \(\int \frac{2}{1 - x^2} \, dx\)

\[ = \int \left( \frac{1}{1 - x} + \frac{1}{1 + x} \right) \, dx \]

\[ = -\log_e(1 - x) + \log_e(1 + x) + c, \quad (-1 < x < 1) \]

\[ = \log_e \left( \frac{1 + x}{1 - x} \right) + c \quad (-1 < x < 1) \]

**b**

1. The degree of the numerator is the same as the degree of the denominator and hence the denominator should divide the numerator using long division.

2. Expand the denominator.

3. Divide the denominator into the numerator.

\[ \frac{x^2 + 6x - 1}{(x + 4)(x + 1)} = \frac{x^2 + 6x - 1}{x^2 + 5x + 4} \]

Using long division:

\[ \begin{array}{c|ccccc}
 & x & + & 1 & 3 \\
\hline
x^2 + 5x + 4 & \frac{1}{x^2 + 6x - 1} & x^2 + 5x + 4 \\
\hline
\end{array} \]

\[ x - 5 \]

The division yields \(1\) with remainder \((x - 5)\).
THINK

4. Rewrite the rational expression using the result of the division.

5. Express \( \frac{x - 5}{(x + 4)(x + 1)} \) as partial fractions with denominators \((x + 4)\) and \((x + 1)\).

6. Rewrite the partial fractions with the original common denominator.

7. Equate the numerators.

8. Let \( x = -1 \) to find \( a \).

9. Solve for \( a \).

10. Let \( x = -4 \) to find \( b \).

11. Solve for \( b \).

12. Express the original integrand in its partial fraction form.

13. Antidifferentiate by rule.

WRITE

Therefore

\[
\frac{x^2 + 6x - 1}{(x + 4)(x + 1)} = 1 + \frac{x - 5}{(x + 4)(x + 1)}
\]

Now

\[
\frac{x - 5}{(x + 4)(x + 1)} = \frac{a}{x + 4} + \frac{b}{x + 1}
\]

\[
= \frac{a(x + 4) + b(x + 1)}{(x + 4)(x + 1)}
\]

and thus \( x - 5 = a(x + 4) + b(x + 1) \)

Let \( x = -1, \ -6 = 3a \)

\[ a = -2 \]

Let \( x = -4, \ -9 = -3b \)

\[ b = 3 \]

Therefore,

\[
\int \frac{x^2 + 6x - 1}{(x + 4)(x + 1)} \, dx
\]

\[
= \int \left( 1 + \frac{-2}{x + 4} + \frac{3}{x + 1} \right) \, dx
\]

\[
= x - 2\log_e(x + 4) + 3\log_e(x + 1) + c,
\]

\((x > -1)\).

Rational polynomials can be antidifferentiated by rewriting the expressions as partial fractions or by long division. If the numerator is of degree less than the denominator then use partial fractions; otherwise rewrite the expression by long division. Two common partial fraction transformations are shown below.

<table>
<thead>
<tr>
<th>Rational expression</th>
<th>Equivalent partial fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{f(x)}{(ax + b)(cx + d)} ) where ( f(x) ) is a linear function</td>
<td>( \frac{A}{(ax + b)} + \frac{B}{(cx + d)} )</td>
</tr>
<tr>
<td>( \frac{f(x)}{(ax + b)^2} ) where ( f(x) ) is a linear function</td>
<td>( \frac{A}{(ax + b)^2} + \frac{B}{(ax + b)} )</td>
</tr>
</tbody>
</table>


**EXERCISE 6D**

**Antidifferentiation using partial fractions**

1. **Find the values of** \( a, b \) **and** \( c \) **in the following identities.**
   - a) \( ax + b(x - 1) = 3x - 2 \)
   - b) \( a(x + 2) + b(x - 3) = x - 8 \)
   - c) \( a(x - 4) + b = 3x - 2 \)
   - d) \( a(3x + 1) + b(x - 2) = 5x + 4 \)
   - e) \( a(2 - 3x) + b(x + 5) = 9x + 11 \)
   - f) \( a(x + 2) + bx = 2x - 10 \)
   - g) \( a + b(x + 2) + c(x + 2)(x + 3) = x^2 + 4x - 2 \)
   - h) \( a(x + 2)(x - 3) + bx(x - 3) + cx(x + 2) = 3x^2 - x + 6 \)

2. **Express each of the following rational expressions as partial fractions.**
   - \( \frac{1}{(x + 1)(x + 2)} \)
   - \( \frac{3x}{(x - 2)(x + 1)} \)
   - \( \frac{4x + 5}{(x + 2)^2} \)
   - \( \frac{7x - 4}{(x - 2)(x + 3)} \)
   - \( \frac{11 - 3x}{(2 - x)(x + 3)} \)
   - \( \frac{6x}{(x + 3)(x - 1)} \)
   - \( \frac{x + 3}{(x + 2)(x + 3)} \)
   - \( \frac{5x - 26}{(x - 5)^2} \)
   - \( \frac{8x - 10}{(2x + 1)(x - 3)} \)
   - \( \frac{12 - 2x}{(1 - x)(3 - x)} \)
   - \( \frac{x + 20}{(x - 4)(x + 4)} \)
   - \( \frac{x + 4}{x(x - 2)} \)
   - \( \frac{9x - 11}{(3x - 2)(x + 1)} \)

3. **Find the antiderivative of each rational expression in question 2.**

4. **multiple choice**
   - a) \( \frac{5x + 10}{24 - 2x - x^2} = \frac{a}{x + 6} + \frac{b}{4 - x} \)
     - A) \( a = 2, b = 3 \)
     - B) \( a = -2, b = 3 \)
     - C) \( a = 3, b = 2 \)
     - D) \( a = -2, b = 3 \)
     - E) \( a = 1, b = -1 \)
   - b) Hence \( \int \frac{5x + 10}{24 - 2x - x^2} \) dx is equal to:
     - A) \( 2\log_e(x + 6) - 3\log_e(4 - x) + c \)
     - B) \( -2\log_e(x + 6) - 3\log_e(4 - x) + c \)
     - C) \( 3\log_e(x + 6) + 2\log_e(4 - x) + c \)
     - D) \( 3\log_e(x + 6) - 2\log_e(4 - x) + c \)
     - E) \( \log_e(x + 6) - \frac{1}{4 - x} \)

5. **multiple choice**
   - The antiderivative of \( \frac{10}{x^2 + x - 6} \) is equal to:
     - A) \( 2\log_e(x + 3) - \log_e(x - 2) + c \)
     - B) \( 2\log_e\frac{x + 1}{x - 6} + c \)
     - C) \( 2\log_e\frac{x + 3}{x - 2} + c \)
     - D) \( \log_e(x + 3) - 2\log_e(x - 2) + c \)
     - E) \( \log_e(x + 1) - 2\log_e(x - 6) + c \)
6 Antidifferentiate each of the following rational polynomials by first expressing them as partial fractions.

a) \( \frac{3x + 10}{x^2 + 2x} \)  

b) \( \frac{5x - 4}{x^2 - x - 2} \)  

c) \( \frac{x + 3}{x^2 + 3x + 2} \)

d) \( \frac{6x - 1}{x^2 - 5x - 6} \)  

e) \( \frac{5x - 7}{x^2 - 4x + 3} \)  

f) \( \frac{x + 16}{x^2 + 7x + 6} \)

g) \( \frac{7x + 9}{x^2 - 9} \)  

h) \( \frac{7x + 1}{x^2 - 1} \)  

i) \( \frac{5x}{2x^2 - 3x - 2} \)

j) \( \frac{16 - 2x}{3x^2 + 7x - 6} \)  

k) \( \frac{x + 4}{2x^2 - 5x + 2} \)  

l) \( \frac{4}{4 - x^2} \)

m) \( \frac{3x - 4}{16 - x^2} \)  

n) \( \frac{x + 13}{5 + 4x - x^2} \)

7 By first simplifying the rational expression using long division, find the antiderivative of each of the following expressions.

a) \( \frac{x - 1}{x + 5} \)  

b) \( \frac{x + 3}{x - 2} \)  

c) \( \frac{x^2 - 1}{x^2 + 3x} \)

d) \( \frac{x^2 + 2x + 4}{x^2 - 4x} \)  

e) \( \frac{x^2 - x}{(x + 3)(x + 1)} \)  

f) \( \frac{x^2 + x + 4}{x^2 - 2x - 3} \)

g) \( \frac{x^2 + 3x - 2}{x^2 - 4} \)  

h) \( \frac{x^3 + 4x^2 - x}{(x + 2)(x + 1)} \)  

i) \( \frac{x^3 + 4x - 13}{x^2 - 4 - x} \)

j) \( \frac{2x^3 + x^2 - 5}{x^2 - 1} \)  

k) \( \frac{x^2 - x + 2}{x^2 + 2x + 1} \)  

l) \( \frac{2x^2 - 9x + 7}{x^2 - 6x + 9} \)

8 Evaluate the following integrals in terms of \( x \).

a) \( \int \frac{4 - x}{x(x + 2)}\,dx \)  

b) \( \int \frac{9x + 8}{(x - 3)(x + 4)}\,dx \)  

c) \( \int \frac{5(x + 1)}{x^2 - 25}\,dx \)

d) \( \int \frac{x^2 + 3}{x^2 - 9}\,dx \)  

e) \( \int \frac{x^2 + 3x - 4}{(x - 4)(x + 2)}\,dx \)  

f) \( \int \frac{x^2 + 4x + 1}{x^2 + 6x - 7}\,dx \)

g) \( \int \frac{x^3 + x^2 - 4x}{x^2 - 4x + 4}\,dx \)  

h) \( \int \frac{4x^2 + 6x - 4}{2x^2 - x - 6}\,dx \)  

i) \( \int \frac{x + 1}{x^2 + 4}\,dx \)

Challenge

k) \( \int \frac{5x^2 + 2x + 17}{(x - 1)(x + 2)(x - 3)}\,dx \)  

l) \( \int \frac{x^2 + 18x + 5}{(x + 1)(x - 2)(x + 3)}\,dx \)

m) \( \int \frac{x^2 + 8x + 9}{(x - 1)(x + 2)^2}\,dx \)  

n) \( \int \frac{x^2 + 5x + 1}{(x + 1)(2 - x)}\,dx \)

9 a) If \( f'(x) = \frac{6}{x^2 - 1} \) and \( f(2) = 3\log_2 2 \), find \( f(x) \).

b) State the domain of \( f(x) \).

10 a) Find \( g(x) \) if \( g'(x) = \frac{x^2 + 1}{x^2 - 2x - 3} \) and \( g(4) = 4 - \log_5 5 \).

b) State the domain of \( g(x) \).
Definite integrals

The quantity \( \int_a^b f(x) \, dx \) is called the ‘definite integral of the function \( f(x) \)’. However, \( \int_a^b f(x) \, dx \) is called the ‘definite integral of the function \( f(x) \)’ and is evaluated using the result that:

\[
\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)
\]

where \( F(x) \) is an antiderivative of \( f(x) \).

The definite integral \( \int_a^b f(x) \, dx \) can be found only if the integrand, \( f(x) \), exists for all values of \( x \) in the interval \([a, b]\); that is, \( a \leq x \leq b \).

WORKED Example 14

For each of the following integrals, state:

\[ \text{i} \quad \text{the domain of the integrand} \quad \text{ii} \quad \text{whether the integral exists.} \]

\[
\begin{align*}
\text{a} & \quad \int_{-2}^{2} \frac{1}{\sqrt{9-x^2}} \, dx \\
\text{b} & \quad \int_{0}^{4} \frac{2}{(x-1)(x+3)} \, dx \\
\end{align*}
\]

THINK

\text{a i} 1 \quad \text{For the integrand to exist,} \sqrt{9-x^2} \text{ must be greater than 0.} \\
2 \quad \text{Solve the inequation for} \ x. \\
3 \quad \text{State the domain.} \\
\text{ii} \quad \text{The integral exists for all values of} \ x \ 	ext{between the terminals} \ -2 \ 	ext{and} \ 2. \\

\text{b i} 1 \quad \text{The integrand does not exist for} \\
x = -3 \ 	ext{and} \ 1, \ 	ext{as these values make the denominator equal to zero.} \\
2 \quad \text{State the domain.} \\
\text{ii} \quad \text{The integral does not exist for all values of} \ x \ 	ext{between the terminals} \ 0 \ 	ext{and} \ 4 \\
\text{(as} \ 1 \ 	ext{lies in the interval).}
\]

WRITE

\text{a i} \quad \text{The integrand exists if} \sqrt{9-x^2} > 0. \\
\quad \quad \quad \quad x^2 < 9 \\
\quad \quad \quad \quad \quad \quad -3 < x < 3 \\
\quad \quad \quad \quad \ 	ext{The domain is} \ (-3, 3). \\
\text{ii} \quad \text{The integral exists.} \\

\text{b i} \quad x \neq -3, 1 \\
\quad \quad \quad \quad \ 	ext{Domain is} \ \mathbb{R}\backslash\{-3, 1\}. \\
\text{ii} \quad \text{The integral does not exist.} 

When using substitution to evaluate definite integrals there is no need to return to an expression in terms of \( x \) providing the terminals are expressed in terms of \( u \). In fact it is mathematically incorrect to show the integral in terms of \( u \) but with terminals in terms of \( x \). Therefore when using a substitution, \( u = f(x) \), the terminals should also be adjusted in terms of \( u \).
Use an appropriate substitution to express each of the following definite integrals in terms of $u$, with the terminals of the integral correctly adjusted.

**a** \( \int_{2}^{3} \frac{x}{x^2 - 1} \, dx \)  

**b** \( \int_{3}^{6} \frac{x}{\sqrt{x - 2}} \, dx \)

**THINK**

**a**  
1. Antidifferentiate the integrand by letting $u = x^2 - 1$ so the derivative method can be applied.

2. Find \( \frac{du}{dx} \).

3. Express $dx$ in terms of $du$.

4. Adjust the terminals by finding $u$ when $x = 2$ and $x = 3$.

5. Rewrite the integral.

6. Simplify the integrand.

**b**  
1. Antidifferentiate the integrand by using the linear substitution $u = x - 2$.

2. Find \( \frac{du}{dx} \).

3. Express $dx$ in terms of $du$.

4. Express $x$ in terms of $u$.

5. Adjust the terminals by finding $u$ when $x = 3$ and $x = 6$.

6. Rewrite the integral.

7. Simplify the integrand.

**WRITE**

**a** Let $u = x^2 - 1$.

\[
\frac{du}{dx} = 2x
\]

or \( dx = \frac{du}{2x} \)

When $x = 2$, $u = 2^2 - 1 = 3$

When $x = 3$, $u = 3^2 - 1 = 8$

Therefore the integral is

\[
\int_{3}^{8} \frac{x}{u} \times \frac{du}{2x}
\]

\[
= \int_{3}^{8} \frac{1}{2u} \, du
\]

**b** Let $u = x - 2$.

\[
\frac{du}{dx} = 1
\]

or \( dx = du \)

$x = u + 2$

When $x = 3$, $u = 3 - 2 = 1$

When $x = 6$, $u = 6 - 2 = 4$

Therefore the integral is

\[
\int_{1}^{4} \frac{u + 2}{u^\frac{1}{2}} \, du
\]

\[
= \int_{1}^{4} \left( u^\frac{1}{2} + 2u^{-\frac{1}{2}} \right) \, du
\]
EXAMPLE 16

Evaluate the following definite integrals.

\( a \int_{0}^{2} \frac{x - 2}{x^2 + 5x + 4} \, dx \quad b \int_{0}^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin x} \, dx \)

**THINK**

**WRITE**

\( a \int_{0}^{2} \frac{x - 2}{x^2 + 5x + 4} \, dx \)

Consider: \( \frac{x - 2}{x^2 + 5x + 4} = \frac{x - 2}{(x + 1)(x + 4)} \)

\( = \frac{a}{x + 1} + \frac{b}{x + 4} \)

\( = \frac{a(x + 4) + b(x + 1)}{x^2 + 5x + 4} \)

\( x - 2 = a(x + 4) + b(x + 1) \)

Let \( x = -1 \) to find \( a \).

\( -3 = 3a \)

\( a = -1 \)

Let \( x = -4 \) to find \( b \).

\( -6 = -3b \)

\( b = 2 \)

So \( \int_{0}^{2} \frac{x - 2}{x^2 + 5x + 4} \, dx \)

\( = \int_{0}^{2} \frac{-1}{x + 1} + \frac{2}{x + 4} \, dx \)

\( = [-\log_e(x + 1) + 2\log_e(x + 4)]_0^2 \)

\( = [-\log_e, 3 + 2\log_e, 6] - [-\log_e, 1 + 2\log_e, 4] \)

\( = -\log_e, 3 + 2\log_e, 6 - 2\log_e, 4 \)

\( = 2\log_e, 1.5 - \log_e, 3 \)

\( = \log_e, 2.25 - \log_e, 3 \)

\( = \log_e, 0.75 \)

\( (\text{or approx. } -2.88) \)

**b**

**WRITE**

\( b \int_{0}^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin x} \, dx \)

Let \( u = 1 + \sin x \)

\( \frac{du}{dx} = \cos x \)

or \( dx = \frac{du}{\cos x} \)
THINK
5 Change terminals by finding $u$ when $x = 0$ and $x = \frac{\pi}{2}$.

WRITE
When $x = 0$, $u = 1 + \sin 0 = 1$
When $x = \frac{\pi}{2}$, $u = 1 + \sin \frac{\pi}{2} = 1 + 1 = 2$

6 Simplify the integrand.

So $\int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin x} \, dx$

$= \int_1^{\frac{1}{2}} (\cos x) u^{\frac{1}{2}} \frac{du}{\cos x}$

$= \int_1^{\frac{1}{2}} u^{\frac{1}{2}} \, du$

7 Antidifferentiate the integrand.

$= \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_{\frac{1}{2}}^{1}$

8 Evaluate the integral.

$= \frac{2}{3} \times 2^\frac{3}{2} - \frac{2}{3} \times 1^\frac{3}{2}$

$= \frac{4\sqrt{2}}{3} - \frac{2}{3}$

or $\frac{4\sqrt{2} - 2}{3}$

WORKED Example 17

By using the substitution $x = \sin \theta$, evaluate $\int_0^{\frac{1}{2}} \sqrt{1 - x^2} \, dx$.

THINK
1 Let $x = \sin \theta$.

2 Find $\frac{dx}{d\theta}$.

3 Make $dx$ the subject.

4 Change the terminals by finding $\theta$ when $x = \frac{1}{2}$ and $x = 0$.

WRITE
Let $x = \sin \theta$.

$\frac{dx}{d\theta} = \cos \theta$

or $dx = \cos \theta \, d\theta$

When $x = \frac{1}{2}$, $\frac{1}{2} = \sin \theta$

$\theta = \frac{\pi}{6}$

When $x = 0$, $0 = \sin \theta$

$\theta = 0$

Continued over page
**THINK**

5. Simplify the integrand.

\[
\int_0^{\frac{1}{2}} \sqrt{1 - x^2} \, dx
\]

\[
= \int_0^{\frac{\pi}{6}} \sqrt{1 - \sin^2 \theta \cos \theta} \, d\theta
\]

\[
= \int_0^{\frac{\pi}{6}} \cos \theta \cos \theta \, d\theta
\]

\[
= \int_0^{\frac{\pi}{6}} \cos^2 \theta \, d\theta
\]

6. Replace \( \cos^2 \theta \) by its identity \( \frac{1}{2} (1 + \cos 2\theta) \).

\[
= \int_0^{\frac{\pi}{6}} \frac{1}{2} (1 + \cos 2\theta) \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\frac{\pi}{6}} (1 + \cos 2\theta) \, d\theta
\]

7. Antidifferentiate the integrand.

\[
= \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{6}}
\]

8. Evaluate the integral.

\[
= \frac{1}{2} \left[ \left( \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} \right) - (0 + \frac{1}{2} \sin 0) \right]
\]

\[
= \frac{1}{2} \left[ \frac{\pi}{6} + \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) \right]
\]

\[
= \frac{\pi}{12} + \frac{\sqrt{3}}{8}
\]

---

**WRITE**

\[
\int_0^{\frac{1}{2}} \sqrt{1 - x^2} \, dx
\]

**Finding the numeric integral at the HOME screen**

To find the value of a definite integral, press \( \text{(MATH)} \) and select \( 9: \text{fnInt}() \). Then type in the integrand, the function variable, the lower terminal and the upper terminal. Press \( \text{ENTER} \) to evaluate the integral.

Alternatively, if the function is already in \( Y_1 \), press \( \text{(MATH)} \), select \( 9: \text{fnInt}() \), complete \( 9: \text{fnInt}(Y_1,X,0,2) \) and press \( \text{ENTER} \). (Remember that to insert the symbol \( Y_1 \), press \( \text{VARS} \), select \( \text{Y–VARS} \) and \( 1: \text{Function} \), then \( 1:Y_1 \) (similarly for any \( Y \) variable).

1. The screen shows both methods for \( \int_0^{\frac{2}{x^2 + 5x + 4}} x - 2 \, dx \) (Worked example 16a).

2. To estimate \( \int_0^{\pi} 2 \cos^2 \frac{x}{2} \, dx \), press \( \text{(MATH)} \), select \( 9: \text{fnInt}() \) and complete by entering \( 2(\cos(X + 2))^2,X,0,\pi) \) and pressing \( \text{ENTER} \).
A handy trick to use, if the answer is a simple fraction, is to press (MATH), select 1: Frac and press (ENTER) — but it doesn’t work in this case.

If the answer could possibly be a fractional multiple of \( \pi \), first try dividing by \( \pi \) then pressing (MATH), selecting 1: Frac and pressing (ENTER). In this case, the answer is just \( \pi \) itself. (Don’t expect this trick to always work!)

Definite integrals

For each of the following definite integrals \( \int_a^b f(x) \, dx \), state i the maximal domain of the integrand \( f(x) \) and ii whether the integral exists.

\[
\begin{align*}
\text{a} & \quad \int_1^2 \frac{1}{9-x^2} \, dx \\
\text{b} & \quad \int_0^1 \frac{-1}{\sqrt{4-x^2}} \, dx \\
\text{c} & \quad \int_3^5 \frac{dx}{\sqrt{16-x^2}} \\
\text{d} & \quad \int_{-1}^2 \frac{dx}{1+x^2} \\
\text{e} & \quad \int_1^4 \frac{2}{x} \, dx \\
\text{f} & \quad \int_1^\infty \frac{dx}{x(x+1)} \\
\text{g} & \quad \int_{-1}^1 \frac{4x+10}{x^2+5x+6} \, dx \\
\text{h} & \quad \int_0^\infty \frac{1}{(x-1)^2} \, dx \\
\text{i} & \quad \int_1^3 \frac{3x+2}{x^2-8x+12} \, dx \\
\text{j} & \quad \int_0^{\sqrt{2}} \frac{dx}{4x^2+9} \\
\text{k} & \quad \int_{-1}^0 \frac{dx}{\sqrt{1-9x^2}} \\
\text{l} & \quad \int_0^1 (2x-1)^3 \, dx \\
\text{m} & \quad \int_0^2 \left( x + \frac{1}{x-2} \right) \, dx \\
\text{n} & \quad \int_0^3 (e^x + e^{-x})^2 \, dx
\end{align*}
\]
2 Evaluate the integrals in question 1 provided that the integrand, \( f(x) \), exists for all values within the domain of the integral.

3 **multiple choice**

The definite integral \( \int_0^2 2x^2 \sqrt{x^3 + 1} \, dx \) can be evaluated after substituting \( u = x^3 + 1 \).

**a** The integral will then be equal to:

A \( \int_0^2 \frac{2\sqrt{u}}{3} \, du \)  
B \( \int_0^9 \frac{3\sqrt{u} + 1}{2} \, du \)  
C \( \int_1^9 \frac{2\sqrt{u}}{3} \, du \)

D \( \int_0^2 2x^2 \sqrt{u} \, du \)  
E \( \int_0^7 \frac{4u^3}{9} \, du \)

**b** The value of the integral is:

A \( 11 \frac{5}{9} \)  
B \( 8\sqrt{2} \)  
C 9  
D \( 12 \frac{4}{9} \)  
E \( 10\sqrt{10} - 1 \)

4 **multiple choice**

**a** \( \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 + \sin x}} \, dx \) can be evaluated by first making the substitution:

A \( u = \sin x \)  
B \( u = \cos x \)  
C \( u = \sqrt{1 + \sin x} \)

D \( u = \cot x \)  
E \( u = 1 + \sin x \)

**b** The integral will then be equal to:

A \( \int_0^1 u^{-\frac{1}{2}} \, du \)  
B \( \int_1^0 u^{\frac{1}{2}} \, du \)  
C \( \int_0^2 u^{-\frac{1}{2}} \, du \)  
D \( \int_1^2 u^{-\frac{1}{2}} \, du \)  
E \( \int_0^1 \sqrt{1 + u} \, du \)

**c** When evaluated, the integral is equal to:

A 2  
B \( 2\sqrt{2} - 2 \)  
C -2  
D \( 2\sqrt{2} \)  
E \( \frac{2}{3} \)

5 By choosing an appropriate substitution for \( u \), express the following integrals in terms of \( u \). (Do not forget to change the terminals.)

**a** \( \int_0^2 x^2(2 + x^3) \, dx \)  
**b** \( \int_1^\sqrt{2} \frac{4x}{x^2 - 3^2} \, dx \)  
**c** \( \int_0^1 x\sqrt{x^2 + 1} \, dx \)

**d** \( \int_2^4 (x - 1)\sqrt{x^2 - 2x} \, dx \)  
**e** \( \int_1^2 x\sqrt{x - 1} \, dx \)  
**f** \( \int_0^3 \frac{x^2}{\sqrt{x + 1}} \, dx \)

**g** \( \int_1^3 \frac{\log_e x}{x} \, dx \)  
**h** \( \int_0^{\frac{\pi}{2}} \sin x \, e^{\cos x} \, dx \)  
**i** \( \int_0^1 x(1 - x)^{10} \, dx \)

**j** \( \int_0^{\frac{\pi}{2}} \cos x\sqrt{\sin x} \, dx \)  
**k** \( \int_0^{\frac{\pi}{4}} \tan^3 x \, \sec^2 x \, dx \)  
**l** \( \int_0^{\frac{\pi}{2}} x \, \sin x^2 \, dx \)

**m** \( \int_0^{\frac{\pi}{2}} \cos^3 x \, dx \)  
**n** \( \int_0^1 \frac{e^x}{\sqrt{x^2 + 1}} \, dx \)

6 Evaluate each of the integrals in question 5.
7 Evaluate the following definite integrals.

\( \int_{-2}^{0} 4xe^{x^2} \, dx \)  
\( \int_{-3}^{2} 2\sqrt{x+3} \, dx \)  
\( \int_{0}^{\pi/3} \sin x \cos^4 x \, dx \)  
\( \int_{-1}^{1} \frac{1}{\sqrt{4-x^2}} \, dx \)  
\( \int_{-1}^{1} \frac{-1}{4-(x-1)^2} \, dx \)  
\( \int_{\pi/4}^{\pi/2} \cot x \, dx \)  
\( \int_{-1}^{1} \frac{2x^2}{x^2+1} \, dx \)  
\( \int_{0}^{\pi} (2x+1)e^{x^2+x} \, dx \)  
\( \int_{2}^{5} \frac{x^2}{\sqrt{x-1}} \, dx \)

8 By substituting \( x = \sin \theta \), evaluate \( \int_{0}^{1} \sqrt{1-x^2} \, dx \).

9 By substituting \( x = 2\sin \theta \), evaluate \( \int_{0}^{\sqrt{3}} \sqrt{4-x^2} \, dx \).

10 By making the substitution \( x = \tan \theta \), evaluate \( \int_{0}^{l} \frac{dx}{(1+x^2)^2} \).

11 If \( \int_{0}^{a} \frac{4}{1+x^2} \, dx = \pi \), find the value of \( a \).

12 If \( \int_{0}^{a} \frac{4}{4-x^2} \, dx = -\log_e 3 \), find \( a \).

13 If \( \int_{-1}^{1} 3\sqrt{x+1} \, dx = 6\sqrt{3} \), find \( a \).

14 If \( \int_{-a}^{a} \frac{1}{\sqrt{4-x^2}} \, dx = \frac{\pi}{2} \), find \( a \).
You can check your answers by using the Mathcad file ‘Integrator’ found on the Maths Quest CD-ROM.

### Applications of integration

In this section, we shall examine how integration may be used to determine the area under a curve and the area between curves.

#### Areas under curves

You will already be aware that the area between a curve which is above the $x$-axis and the $x$-axis itself is as shown in the diagram at right.

$$\text{Area} = \int_a^b f(x) \,dx$$

Further, the area between a curve which is below the $x$-axis, and the $x$-axis itself, is as shown in the second diagram.

$$\text{Area} = -\int_a^b g(x) \,dx$$

$$= \left| \int_a^b g(x) \,dx \right|$$

The modulus is required here since, for a curve segment that lies below the $x$-axis, the integral associated with that curve segment is a negative number. Area is a positive number and in this case the integral is negative.
Similarly, the area between a curve and the $y$-axis can be found if the rule for the curve is expressed as a function of $y$, that is, $x = f(y)$.

$$\text{Area} = \int_{a}^{b} f(y) \, dy$$  \hspace{0.5cm} \text{(integral measures to the right of the $y$-axis are positive)}

or

$$\text{Area} = -\int_{a}^{b} g(y) \, dy$$  \hspace{0.5cm} \text{(integral measures to the left of the $y$-axis are negative)}

$$= \left| \int_{a}^{b} g(y) \, dy \right|$$

If the graph crosses the $x$-axis, then the areas of the regions above and below the $x$-axis have to be calculated separately. In this case the $x$-intercepts must be determined. In the figure at right a single intercept, $c$, is shown.

$$\text{Area} = \int_{c}^{b} f(x) \, dx + \int_{a}^{c} f(x) \, dx$$

$$= \int_{c}^{b} f(x) \, dx - \int_{a}^{c} f(x) \, dx$$

Similarly the shaded region in the figure at right has an area given by:

$$\text{Area} = \int_{c}^{b} f(y) \, dy + \int_{a}^{c} f(y) \, dy$$

$$= \int_{c}^{b} f(y) \, dy - \int_{a}^{c} f(y) \, dy$$

**WORKED Example 18**

If $y = \frac{2 \log_e x}{x}$, find:

- **a** the $x$-intercepts
- **b** the area bounded by the curve, the $x$-axis and the line $x = 3$.

**THINK**

- **a** 1. For $x$-intercepts, $y = 0$, when $2\log_e x = 0$.
  2. Solve for $x$.
- **b** 1. Sketch a graph showing the region required. (A graphics calculator may be used.)

**WRITE**

- **a** $x$-intercepts occur when $2\log_e x = 0$.
  That is, $x = 1$.
- **b** Sketch a graph showing the region required.
Express the area as a definite integral.

Antidifferentiate by letting $u = \log_e x$ to apply the derivative method.

Find $\frac{du}{dx}$.

Make $dx$ the subject.

Express the terminals in terms of $u$.

Simplify the integrand.

Antidifferentiate the integrand.

Evaluate the integral.

State the area.

A graphics calculator should be used here to verify the result.

Finding the numeric integral at the GRAPH screen

To find the area under a curve between two $x$-values, first graph the curve by entering its equation as $Y_1$ in the $Y=$ menu.

Consider $y = \frac{2\log_e x}{x}$ in worked example 18. Press $Y=$ and type in $(2\ln(X)) + X$ at $Y_1$.

Then press (GRAPH).

To find the area bounded by this curve and the $x$-axis between $x = 1$ and $x = 3$, press (2nd) [CALC] and select 7: $\int f(x) \, dx$. Type in 1 for the lower value (press ENTER) and 3 for the upper value (press ENTER). Compare this result to that obtained in worked example 18.
WORKED Example 19

Examine the figure at right.

a Express the rule as a function of $y$.

b Find the area of the shaded section.

THINK

a 1 Write down the rule.
   2 Square both sides of the equation.
   3 Add 1 to both sides to make $x$ the subject.

b 1 Express the area between the curve and the $y$-axis in integral notation.
   2 Antidifferentiate by rule.
   3 Evaluate the integral.

WRITE

a $y = \sqrt{x - 1}$
   $y^2 = x - 1$
   or $x = y^2 + 1$

b Area $= \int_{0}^{2} (y^2 + 1) \, dy$
   $= \left[ \frac{1}{3}y^3 + y \right]_0^2$
   $= \left[ \frac{8}{3} + 2 \right] - [0 + 0]$
   $= 4 \frac{2}{3}$

The area is $4 \frac{2}{3}$ square units.

Using symmetry properties

In some problems involving area calculations, use of symmetry properties can simplify the procedure.

WORKED Example 20

Find the area inside the ellipse in the figure at right.

THINK

1 Write the equation. (The ellipse is symmetrical about the $x$-axis and $y$-axis and so finding the shaded area in the figure allows for the total enclosed area to be determined.)

2 Express the relation as a function of $x$ for the top half of the ellipse.

WRITE

$x^2 + \frac{y^2}{4} = 1$

$\frac{y^2}{4} = 1 - x^2$

$y^2 = 4(1 - x^2)$

$y = 2\sqrt{1 - x^2}$ is the rule for the top half of the ellipse.

$(y = -2\sqrt{1 - x^2}$ is the bottom half.)

Continued over page
Areas between curves

When finding the areas between two curves that intersect, it is necessary to determine where the point of intersection occurs. In the figure at right, two functions, \( f \) and \( g \), intersect at the point \( P \) with \( x \)-ordinate \( c \).

The area contained within the envelope of the two functions bounded by \( x = a \) and \( x = b \) is given by:

\[
\int_{a}^{b} [f(x) - g(x)] \, dx
\]
Similarly, areas between curves can also be found relative to the y-axis.

\[
\text{Area } = \int_a^b [g(y) - f(y)] \, dy
\]

Note that on the interval \([a, b]\), \(g(y) \geq f(y)\) and hence the integrand is \(g(y) - f(y)\) and not \(f(y) - g(y)\).

When an area between a curve and the x-axis (or between curves) gives an integrand which cannot be antidifferentiated, it may be possible to express the area relative to the y-axis, creating an integrand which can be antidifferentiated.

**WORKED Example 21**

Find the area bounded by the curves \(y = x^2 - 2\) and \(y = 2x + 1\).

**THINK**

1. Check on a graphics calculator to see if the curves intersect. If they do, solve \(x^2 - 2 = 2x + 1\) to find the x-ordinate of the point or points of intersection for the two curves.

2. Express the area as an integral. (Use \(\int \), as without a graph we cannot always be sure which function is above the other. Here is a valuable use for the graphics calculator.)

**WRITE**

\[
\text{Area } = \left| \int_{-1}^3 [x^2 - 2 - (2x + 1)] \, dx \right|
\]

3. Simplify the integral.

\[
= \left| \int_{-1}^3 (x^2 - 2x - 3) \, dx \right|
\]

4. Antidifferentiate by rule.

\[
= \left| \left[ \frac{1}{3}x^3 - x^2 - 3x \right]_{-1}^3 \right|
\]

5. Evaluate the integral.

\[
= \left| \left[ \left(9 - 9 - 9\right) - \left(-\frac{1}{3} - 1 + 3\right)\right] \right|
\]

\[
= \left| -9 - 1 \frac{2}{3} \right|
\]

\[
= \left| -10 \frac{2}{3} \right|
\]

\[
= 10 \frac{2}{3}
\]

6. State the solution.

The area bounded by the two curves is \(10 \frac{2}{3}\) square units.
Consider using the TI calculator for worked example 21.

1. To graph the two curves with equations \( y = x^2 - 2 \) and \( y = 2x + 1 \) enter \( Y_1 = X^2 - 2 \) and \( Y_2 = 2X + 1 \). Then press [GRAPH]. Use TRACE to locate the points of intersection. Adjust the WINDOW settings if necessary.

2. To show the area bounded by the two curves, press [Y=], position the cursor to the left of the \( Y_1 \) symbol and press [ENTER] successively to obtain the ‘shade below’ style. Repeat for \( Y_2 \) to obtain the ‘shade above’ style. Press [GRAPH]. The required area is shown unshaded.

3. To determine the value of the area bounded by the curves on the required interval (in this case, between \( x = -1 \) and \( x = 3 \)), press [MATH], select 9 and complete 9: \( \text{fnInt}(Y_2 - Y_1, X, -1, 3) \) and press [ENTER]. Remember, to insert \( Y_1 \), press [VARS] and select \( Y-VARS, 1: \text{Function} \) and 1:Y1 (or 2:Y2 to enter Y2). Note that in this case we are subtracting \( Y_1 \) from \( Y_2 \) (seen by viewing the graph). However, if it is entered the opposite way, it only produces the negative of the required answer.

---

### Graphics Calculator tip!

**Showing and finding the area bounded by two curves**

1. The area between a curve \( f(x) \), the \( x \)-axis and lines \( x = a \) and \( x = b \) is given by:
   \[
   \text{Area} = \int_a^b f(x) \, dx = F(b) - F(a) \quad \text{where} \quad F(x) \text{ is the antiderivative of } f(x).
   \]

2. Area measures can also be evaluated by integration along the \( y \)-axis. The area between a curve \( f(y) \), the \( y \)-axis and lines \( y = a \) and \( y = b \) is given by:
   \[
   \text{Area} = \int_a^b f(y) \, dy = F(b) - F(a) \quad \text{where} \quad F(y) \text{ is the antiderivative of } f(y).
   \]

3. If an area measure is to be evaluated over the interval \([a, b]\) and the curve crosses the \( x \)-axis at \( x = c \) between \( a \) and \( b \), then the integral has to be decomposed into two portions.
   \[
   \text{Area} = \left| \int_a^c f(x) \, dx \right| + \left| \int_c^b f(x) \, dx \right| = |F(c) - F(a)| + |F(b) - F(c)|
   \]

4. The area bounded by two curves \( f(x) \) and \( g(x) \) where \( f(x) \geq g(x) \) and the lines \( x = a \) and \( x = b \) is given by:
   \[
   \text{Area} = \int_a^b \left[ f(x) - g(x) \right] \, dx = F(b) - G(b) - F(a) + G(a)
   \]

5. Where possible use a graphics calculator to draw the function or functions to determine whether the integrals have to be decomposed into portions and to check and verify the correct use of the modulus function.
EXERCISE 6F

Applications of integration

For the following problems, give exact answers wherever possible; otherwise give answers to an appropriate number of decimal places. (Use a graphics calculator to assist with, or verify, any graphing required.)

1 For each of the following curves find:
   i the x-intercepts
   ii the area between the curve, the x-axis and the given lines.

   a \( y = \sqrt{x}, \ x = 0 \) and \( x = 9 \)
   b \( y = x - \frac{1}{x^2}, \ x = 1 \) and \( x = 2 \)
   c \( y = x\sqrt{x-1}, \ x = 2 \) and \( x = 5 \)
   d \( y = \frac{3x-2}{x^2-4}, \ x = 3 \) and \( x = 4 \)
   e \( y = \frac{1}{\sqrt{4-x^2}}, \ x = 1 \) and \( x = \sqrt{3} \)
   f \( y = \cos^2x, \ x = 0 \) and \( x = \frac{\pi}{2} \)
   g \( y = 2x \cos x^2, \ x = -\frac{\pi}{3} \) and \( x = 0 \)
   h \( y = \frac{e^x}{2 + e^x}, \ x = 0 \) and \( x = 1 \)

2 For each of the graphs below:
   i express the relationship as a function of y (that is, make x the subject of the rule)
   ii find the magnitude of the shaded area between the curve and the y-axis.

   a
   b
   c
   d
   e
   f
   g
3 Find the magnitude of the shaded areas on each graph below.

a

\[ y = x^2 \]

b

\[ y^2 = x \]

c

\[ y = \frac{1}{2} x^2 \]

\[ x = 0 \]

\[ x = 2 \]

\[ y = 0 \]

\[ y = 2 \]

\[ x = 0 \]

\[ x = 1 \]

\[ y = 0 \]

\[ y = 1 \]

\[ x = 0 \]

\[ x = 1 \]

\[ y = 0 \]

\[ y = \sin^3 x \]

4 multiple choice

a The definite integral that correctly gives the area bounded by the curve \( y = 4x - x^2 \) and the x-axis is:

- A \( \int_0^2 (4x - x^2) \, dx \)
- B \( \int_0^1 (4x - x^2) \, dx \)
- C \( \int_0^4 (4x - x^2) \, dx \)
- D \( \int_0^2 (4x - x^2) \, dx \)
- E \( \int_0^2 (2x^2 - \frac{1}{3}x^3) \, dx \)

b The area, in square units, is equal to:

- A \( \frac{10}{3} \)
- B \( \frac{2}{3} \)
- C \( \frac{5}{3} \)
- D \( 8 \)
- E \( -\frac{5}{3} \)

5 multiple choice

a Which of the graphs below correctly shows the area bounded by the curve \( y^2 = x + 1 \) and the y-axis?

- A

- B

- C

- D

- E

b The definite integral which gives the area bounded by \( y^2 = x + 1 \) and the y-axis is:

- A \( \int_0^1 (y^2 - 1) \, dy \)
- B \( \int_0^0 (y^2 - 1) \, dy \)
- C \( 2 \int_0^0 (y^2 - 1) \, dy \)
- D \( \int_0^1 (y^2 + 1) \, dy \)
- E \( \int_0^1 \sqrt{x - 1} \, dx \)
c The value of the area, in square units, is equal to

\[
\begin{array}{cccccc}
A & \frac{2}{3} & B & \frac{2}{3} & C & \frac{1}{3} & D & \frac{5}{3} & E & 2
\end{array}
\]

6 Find the area bounded by the graph with equation \(y = (x - 2)^2(x + 1)\) and the x-axis.

7 Find the area bounded by the graph with equation \(y^2 = x + 4\) and the y-axis.

8 a Show that the graphs of \(f(x) = x^2 - 4\) and \(g(x) = 4 - x^2\) intersect at \(x = -2\) and \(x = 2\).
   b Find the area bounded by the graphs of \(f(x)\) and \(g(x)\).

9 a On the same axis sketch the graphs of \(f(x) = \sin x\) and \(g(x) = \cos x\) over \([0, \pi]\).
   b Show algebraically that the graphs intersect at \(x = \frac{\pi}{4}\).
   c Find the area bounded by the curves and the y-axis.

10 a On the same axis sketch the graphs of \(y = \sqrt{9 - x}\) and \(y = x + 3\).
   b Find the value of \(x\) where the graphs intersect.
   c Hence find the area between the curves from \(x = -1\) to \(x = 2\).

11 Find the area bounded by the curves \(y = x^2\) and \(y = 3x + 4\).

12 Find the area enclosed by the curves \(y = x^2\) and \(y = \sqrt{x}\).

13 Find the area bounded by \(y = e^x\) and \(y = e^{-x}\) and the line \(y = e\).

14 Examine the figure at right.
   a Find the area enclosed by \(f(x)\), \(g(x)\) and the y-axis.
   b Find the shaded area.

15 Find the area of the ellipse with equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\).
   Hints: 1. Use symmetry with properties.
          2. Antidifferentiate \(\sqrt{a^2 - x^2}\) by using the substitution \(x = a \sin \theta\).

16 Find the area between the circle \(x^2 + y^2 = 9\) and ellipse \(\frac{x^2}{9} + \frac{y^2}{4} = 1\).
   Hint: Make use of symmetry properties.

17 a Sketch the curve \(y = e^{x^2 + 2}\).
   b Find the equation of the tangent at \(x = -2\).
   c Find the area between the curve, the tangent and the y-axis.

18 a Sketch the graph of \(y = \frac{1 - x}{x + 1}\).
   b Find the area bounded by this curve and the x- and y-axes.

19 a Show algebraically that the line \(y = x\) does not meet the curve \(y = \frac{1}{\sqrt{1 - x^2}}\).
   b Find the area enclosed by the curve, the lines \(y = x\) and \(x = \frac{1}{\sqrt{2}}\), and the y-axis.
Volumes of solids of revolution

If part of a curve is rotated about the $x$-axis, or $y$-axis, a figure called a solid of revolution is formed. For example, a solid of revolution is obtained if the shaded region in figure 1 is rotated about the $x$-axis.

The solid generated (figure 2) is symmetrical about the $x$-axis and any vertical cross-section is circular, with a radius equal to the value of $y$ at that point. For example, the radius at $x = a$ is $f(a)$.

Any thin vertical slice may be considered to be cylindrical, with radius $y$ and height $\delta x$ (figure 3).

The volume of the solid of revolution generated between $x = a$ and $x = b$ is found by allowing the height of each cylinder, $\delta x$, to be as small as possible and adding the volumes of all of the cylinders formed between $x = a$ and $x = b$. That is, the volume of a typical strip is equal to $\pi y^2 \delta x$.

Therefore the volume of the solid contained from $x = a$ to $x = b$ is the sum of all the infinitesimal volumes:

$$V = \lim_{\delta x \to 0} \sum_{x = a}^{x = b} \pi y^2 \delta x$$

$$= \int_{a}^{b} \pi y^2 \, dx$$

The value of $y$ must be expressed in terms of $x$ so that the integral can be evaluated. From the figure above $y = f(x)$ and thus the volume of revolution of a curve $f(x)$ from $x = a$ to $x = b$ is $V = \pi \int_{a}^{b} [f(x)]^2 \, dx$.

Similarly if a curve is rotated about the $y$-axis, the solid of revolution shown in the figure at right is produced.

The volume of the solid of revolution is likewise

$$V = \pi \int_{a}^{b} [f(y)]^2 \, dy$$

For regions between two curves that are rotated about the $x$-axis:

$$V = \pi \int_{a}^{b} [f(x)]^2 - [g(x)]^2 \, dx$$
WORKED Example 22

a Sketch the graph of \( y = 2x \) and show the region bounded by the graph, the \( x \)-axis and the line \( x = 2 \).

b Find the volume of the solid of revolution when the region is rotated about the \( x \)-axis.

**THINK**

a 1 Sketch the graph.
2 Shade the region required.

b 1 State the integral that gives the volume. (The volume generated is bounded by \( x = 0 \) and \( x = 2 \)).
2 Simplify the integrand.
3 Antidifferentiate by rule.
4 Evaluate the integral.
5 State the volume.

**WRITE**

\[ V = \pi \int_0^2 (2x)^2 \, dx \]

\[ = \pi \int_0^2 4x^2 \, dx \]

\[ = \pi \left[ \frac{4}{3}x^3 \right]_0^2 \]

\[ = \pi \left[ \frac{32}{3} - 0 \right] \]

\[ = \frac{32\pi}{3} \]

The exact volume generated is \( \frac{32\pi}{3} \) cubic units.

**Graphics Calculator tip!**

Finding the volume of a solid of revolution

Consider the volume of the solid of revolution formed in Worked example 22.

1. The line with equation \( y = 2x \) is rotated about the \( x \)-axis to form a cone. To graph the line, enter \( Y_1 = 2X \) and press \( \text{GRAPH} \). Press \( \text{TRACE} \) to locate particular coordinates.

2. To determine the volume of the cone, press \( \text{MATH} \), then select \( 9: \text{fnInt} \) and insert \( \pi Y_1^2, X, 0, 2 \) and press \( \text{ENTER} \). To insert \( Y_1 \), press \( \text{VARS} \) and select \( \text{Y-VARS}, 1: \text{Function} \) and \( 1: Y_1 \) (similarly any \( Y \) variable).

(Note that \( \pi Y_1^2 \) provides the integrand, \( X \) is the variable, 0 and 2 are the terminals of the integral.)

Can you verify the formula \( V = \frac{1}{3} \pi r^3 \) for this cone? What is the radius for this cone?

You can try to convert your volume answer to a fraction of \( \pi \). Press \( \pm \) and \( 2nd \) [\( \pi \)], then \( \text{MATH} \). Select \( 1: \text{Frac} \) and press \( \text{ENTER} \). This is also shown in the screen above.
Sketch the region bounded by the curve \( y = \log_e x \), the \( x \)-axis, the \( y \)-axis and the line \( y = 2 \).

Calculate the volume of the solid generated if the region is rotated about the \( y \)-axis.

**THINK**

1. Sketch the graph. (Use a graphics calculator if necessary.)
2. Shade the region required.

**WRITE**

\[
\begin{align*}
&\text{a} \quad y = \log_e x \\
&\text{b} \quad y = \log_e x \\
&\quad e^y = e^{\log_e x} \\
&\quad e^y = x \\
&\quad \text{or } x = e^y \\
&\text{So } V = \pi \int_0^2 (e^y)^2 \, dy \\
&\quad = \pi \int_0^2 e^{2y} \, dy \\
&\quad = \pi \left[ \frac{1}{2} e^{2y} \right]_0^2 \\
&\quad = \pi \left[ \frac{1}{2} e^4 - \frac{1}{2} e^0 \right] \\
&\quad = \frac{\pi}{2} (e^4 - 1) \\
&\text{The volume is exactly } \frac{\pi}{2} (e^4 - 1) \text{ cubic units} \\
&\quad \text{(or approximately 84.19 cubic units).}
\end{align*}
\]

**remember**

1. To find the volume of revolution about the \( x \)-axis for the function \( f(x) \) from \( x = a \) to \( x = b \), evaluate the integral:
\[
V = \pi \int_a^b [f(x)]^2 \, dx
\]
2. To find the volume of revolution about the \( y \)-axis for the function \( f(y) \) from \( y = a \) to \( y = b \), evaluate the integral:
\[
V = \pi \int_a^b [f(y)]^2 \, dy
\]
3. To find the volume of revolution about the \( x \)-axis for the region between \( f(x) \) and \( g(x) \) where \( f(x) \geq g(x) \) from \( x = a \) to \( x = b \), evaluate the integral:
\[
V = \pi \int_a^b [f(x)]^2 - [g(x)]^2 \, dx
\]
Volumes of solids of revolution

Give exact answers where possible; otherwise use an appropriate number of decimal places when giving approximate answers. (Use a graphics calculator to check any graphing.)

1. a Sketch the graph of the region bounded by the x-axis, the curve \( y = 3x \) and the line \( x = 2 \).
   b Calculate the volume generated by rotating this region about the x-axis.
   c Verify this result by using the standard volume formula for the solid generated.

2. The region bounded by the graph of \( y = \sqrt{16 - x^2} \) and the x-axis is rotated about the x-axis.
   a Calculate the volume of the solid of revolution generated.
   b Verify this answer using the standard volume formula.

3. a Sketch the region bounded by the curve \( y = \sqrt{x - 1} \), the y-axis and the lines \( y = 0 \) and \( y = 2 \).
   b Calculate the volume generated when this region is rotated about the y-axis.

4. Find the volume generated when the area bounded by \( y = x^2 - 1 \) and the x-axis is rotated about:
   a the x-axis
   b the y-axis.

5. For the regions bounded by the x-axis, the following curves, and the given lines:
   i sketch a graph shading the region
   ii find the volume generated when the region is rotated about the x-axis.
   a \( y = x + 1; \ x = 0 \) and \( x = 2 \)
   b \( y = \sqrt{x}; \ x = 1 \) and \( x = 4 \)
   c \( y = x^2; \ x = 0 \) and \( x = 2 \)
   d \( y^2 = 2x + 1; \ x = 0 \) and \( x = 3 \)
   e \( x^2 + y^2 = 4; \ x = -1 \) and \( x = 1 \)
   f \( y = \frac{2}{x}; \ x = 1 \) and \( x = 3 \)
   g \( y = \cos x; \ x = -\frac{\pi}{2} \) and \( x = \frac{\pi}{2} \)
   h \( y = e^x + 1; \ x = -2 \) and \( x = -1 \)

6. For each region defined in question 5 (a to f only) find the volume generated by rotating it about the y-axis.

7. **Multiple choice**
   a The region bounded by the curves \( y = x^2 + 2 \) and \( y = 4 - x^2 \) is represented by the graph:
      A
      B
      C
      D
      E
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b The volume generated when the region is rotated about the x-axis is equal to:

A \( \pi \int_0^1 (2 - 2x^2)^2 \, dx \)  
B \( \pi \int_{-1}^{1} (2 - 2y)^2 \, dy \)  
C \( \pi \int_0^1 (2 - 2x^2)^2 \, dx \)  
D \( \pi \int_{-1}^{1} (2 - 2x^2)^2 \, dx \)  
E \( \pi \int_{-1}^{1} (6 - 2x^2)^2 \, dx \)

c The volume generated when the region is rotated about the y-axis is equal to:

A \( \pi \int_{3}^{4} (4 - y) \, dy + \pi \int_{3}^{2} (y - 2) \, dy \)  
B \( \pi \int_{3}^{4} (y - 2) \, dy + \pi \int_{3}^{2} (4 - y) \, dy \)  
C \( \pi \int_{2}^{4} (2 - 2y) \, dy \)  
D \( \pi \int_{2}^{4} (2y - 2) \, dy \)  
E \( \pi \int_{2}^{4} (2 - 2x^2) \, dx \)

8 Find the volume generated when the region bounded by the curves \( y = x^2 \) and \( y = -x \) is rotated about:

a the x-axis  
b the y-axis.

9 Find the volume generated when the area bounded by the curve \( y = \sec x \), the line \( x = \frac{\pi}{4} \) and the x- and y-axes is rotated about the x-axis.

10 Find the volume generated by rotating the area bounded by \( y = e^{2x} \), the y-axis and the line \( y = 2 \) about the x-axis.

11 The area bounded by the curve \( y = \tan^{-1}x \), the x-axis and the line \( x = 1 \) is rotated about the y-axis. Find the volume of the solid generated.

12 A model for a container is formed by rotating the area under the curve of \( y = 2 - \frac{x^2}{6} \) between \( x = -1 \) and \( x = 1 \) about the x-axis. Find the volume of the container.
13 For the graph shown at right:
   a find the coordinate of A  
   b find the volume generated when the shaded region is rotated about the x-axis  
   c find the volume generated when the shaded region is rotated about the y-axis.

14 What is the volume generated by rotating the ellipse with equation \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) about:
   a the x-axis?
   b the y-axis?

15 Find the volume generated when the region bounded by \( y = x^2 \) and \( y = \sqrt{8x} \) is rotated about:
   a the x-axis  
   b the y-axis.

16 Find the volume generated by the rotation of the area bounded by the curves \( y = x^3 \) and \( y = x^2 \) about:
   a the x-axis  
   b the y-axis.

17 A hemispherical bowl of radius 10 cm contains water to a depth of 5 cm. What is the volume of water in the bowl?

18 A solid sphere of radius 6 cm has a cylindrical hole of radius 1 cm bored through its centre. What is the volume of the remainder of the sphere?

19 Find the volume of a truncated cone of height 10 cm, a base radius of 5 cm and a top radius of 2 cm.

20 a Find the equation of the circle sketched below.
   b Find the volume of a torus (doughnut-shaped figure) generated by rotating this circle about the x-axis (give your answer in cm\(^3\)).
Approximate evaluation of definite integrals and areas

When calculating definite integrals or areas that involve integrands which cannot be antidifferentiated using techniques discussed in this chapter, approximation methods can be used. We shall now look at two useful and simple approximation methods: the midpoint rule and the trapezoidal rule.

The midpoint rule

The definite integral \( \int_a^b f(x) \, dx \) determines the shaded area under the curve below.

It can be approximated by constructing a rectangle with height equal to the value of \( y \) halfway between \( x = a \) and \( x = b \).

Area of rectangle = \((b - a)f\left(\frac{a + b}{2}\right)\)

The estimate for the shaded area is improved by increasing the number of intervals, that is the number of rectangles between \( x = a \) and \( x = b \). In the figure below, the region from \( x = a \) and \( x = b \) is broken up into \( n \) rectangles.

The base width of each rectangle is \( \delta x \) and the height of each individual rectangle is obtained from the midpoint rule. The area of each rectangle is given by the product of the height and the common width \( \delta x \).

So \( \int_a^b f(x) \, dx \approx \delta x \left[ f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \ldots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right] \)

where: \( \delta x = \frac{b-a}{n} \), the width of each rectangle

\( n = \) the number of intervals and hence rectangles used

\( x_0 = a \)

\( x_n = b \)

WORKED Example 24

Estimate \( \int_0^4 (x^2 + 2x) \, dx \) using the midpoint rule and 4 intervals.

THINK

1. State \( f(x) \).
2. Calculate \( \delta x \).

WRITE

\( f(x) = x^2 + 2x \)

\( \delta x = \frac{4 - 0}{4} = 1 \)
The trapezoidal rule

The area under a curve can also be approximated using a trapezium.

The area of the trapezium

\[ \text{area} = \frac{h}{2} [f(a) + f(b)] \]

By increasing the number of intervals between \( x = a \) and \( x = b \), that is, the number of trapezia, the estimate becomes more accurate:

\[ \int_a^b f(x) \, dx \approx \frac{b - a}{2n} \sum_{k=1}^{n} [f(x_{k-1}) + f(x_k)] \]

Notice here that the terms \( f(a) \) and \( f(b) \) occur only once and all other terms such as \( f(x_1) \) and \( f(x_2) \) occur twice. Thus an approximation to the area is:

\[ \text{Approximate area} = \frac{\delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)] \]

where: \( n \) = the number of intervals used

\[ \delta x = \frac{b - a}{n} \]
\[ a = x_0 \]
\[ b = x_n \]
Estimate \( \int_{0}^{4} (x^2 + 2x) \, dx \) using the trapezoidal rule and four equal intervals.

**THINK**

1. State \( f(x) \).
   \( f(x) = x^2 + 2x \)

2. Calculate \( \delta x \).
   \( \delta x = \frac{4 - 0}{4} = 1 \)

3. Find \( x_0, x_1, x_2, x_3, x_4 \).
   \( x_0 = 0 \)
   \( x_1 = 1 \)
   \( x_2 = 2 \)
   \( x_3 = 3 \)
   \( x_4 = 4 \)

4. Substitute these values into the trapezoidal rule.
   So \( \int_{0}^{4} (x^2 + 2x) \, dx \)
   \( = \frac{1}{2} [f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] \)

5. Evaluate the approximation.
   \( = \frac{1}{2} [0 + 2(3) + 2(8) + 2(15) + 24] \)
   \( = \frac{1}{2} (76) \)
   \( = 38 \)

6. State the solution.
   The value of the definite integral is approximately 38.

Compare this answer with that in worked example 24. Which is closest to the exact answer?

**WORKED Example 26**

Estimate the area under the graph of \( y = x \log_e x \) from \( x = 1 \) to \( x = 5 \) using two equal intervals and:

**a** the midpoint rule  
**b** the trapezoidal rule.

**THINK**

**a** 1. State \( f(x) \).
   \( f(x) = x \log_e x \)

2. Calculate \( \delta x \).
   \( \delta x = \frac{5 - 1}{2} = 2 \)

3. Find \( x_0, x_1, x_2 \).
   \( x_0 = 1 \)
   \( x_1 = 3 \)
   \( x_2 = 5 \)

4. Substitute the values into the midpoint rule.
   So the area \( = \int_{1}^{5} x \log_e x \, dx \)
   \( \approx 2[f(2) + f(4)] \)
   \( = 2[2\log_e 2 + 4\log_e 4] \)
   \( = 4\log_e 2 + 8\log_e 4 \)

5. Evaluate the estimate of the area.

6. State the approximate area.
   The approximate area is 13.863 square units.
**THINK**

b 1. State $f(x)$.

b 2. Calculate $\delta x$.

b 3. Find $x_0$, $x_1$, $x_2$.

b 4. Substitute these values into the trapezoidal rule.

b 5. Evaluate the estimate for the area.

b 6. State the approximate area.

**WRITE**

b $f(x) = x\log_e x$

\[
\delta x = \frac{5-1}{2} = 2
\]

$x_0 = 1$

$x_1 = 3$

$x_2 = 5$

So the area $= \int_1^5 x\log_e x \, dx$

\[
\approx \frac{2}{2} [f(1) + 2f(3) + f(5)]
\]

$= 1[\log_e 1 + 2(3\log_e 3) + 5 \log_e 5]$

$= 6 \log_e 3 + 5 \log_e 5$

The approximate area is 14.639 square units.

---

**remember**

1. Some functions cannot be integrated using the techniques covered in this chapter. Two approximation methods are discussed. The midpoint rule involves subdividing the required area into a finite number of rectangles. The trapezium rule involves subdividing the area into a finite number of trapezia.

2. The midpoint rule:

\[
\int_a^b f(x) \, dx \approx \delta x \left[ f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \ldots + f\left(\frac{x_{n-1} + x_n}{2}\right) \right]
\]

where: $\delta x = \frac{b-a}{n}$, the width of each rectangle

$n = $ the number of intervals used

$x_0 = a$

$x_n = b$

3. The trapezoidal rule:

\[
\int_a^b f(x) \, dx \approx \frac{\delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n) \right]
\]

where: $n = $ the number of intervals used

$\delta x = \frac{b-a}{n}$

$a = x_0$

$b = x_n$
1 Find approximations to the following definite integrals using the midpoint rule with four equal intervals.
   a \( \int_{3}^{5} \frac{dx}{x-2} \)
   b \( \int_{0}^{\pi} \sin x \, dx \)
   c \( \int_{-3}^{-1} \log_{e} x^2 \, dx \)
   d \( \int_{0}^{4} \tan^{-1} x \, dx \)

2 Repeat question 1 using the trapezoidal rule.

3 Use the midpoint rule with two equal intervals to estimate the following definite integrals.
   a \( \int_{0}^{2} (x + 2) \, dx \)
   b \( \int_{1}^{4} (x^2 - 3) \, dx \)
   c \( \int_{1}^{2} (x^3 - x^2 + 2x) \, dx \)
   d \( \int_{0}^{16} (16 - x^2) \, dx \)

4 Repeat question 3 using the trapezoidal rule.

5 **multiple choice**
   a Using the midpoint rule and two equal intervals, an estimate for \( \int_{0}^{1} \frac{1}{1 + x^2} \, dx \) is:
     A 1.4     B 0.8     C 0.9412     D 0.7906     E 0.863
   b Compared to the exact answer, the percentage error in answer a is closest to:
     A 0.7     B 1.9     C 2.5     D 20     E 6.4

6 **multiple choice**
   a Using the trapezoidal rule and four equal intervals, an estimate for \( \int_{0}^{\pi/2} \cos x \, dx \) is:
     A \( 1 + \sqrt{2} \)     B \( \frac{\pi}{2} \)     C \( \frac{\pi(1 + \sqrt{2})}{4} \)     D \( \frac{\pi(1 + \sqrt{2})}{8} \)     E \( \frac{\pi}{4} \)
   b The percentage error relative to the exact answer is closest to:
     A 61     B 21     C 11     D 5     E 1

7 Find the value of \( \int_{1}^{5} e^x \, dx \) using the midpoint rule and:
   a 4 equal intervals
   b 8 equal intervals.
8 Find \( \int_0^4 \sqrt{1 + x^2} \, dx \) using the trapezoidal rule and:
   a) 4 equal intervals
   b) 8 equal intervals.

9 Estimate the area under the curve \( y = \log_e x \) from \( x = 1 \) to \( x = 4 \) using the midpoint rule and:
   a) 3 equal intervals
   b) 6 equal intervals.

10 Estimate the area under the graph of \( y = \cos^{-1} x \) from \( x = -1 \) to \( x = 1 \) using the midpoint rule and 4 equal intervals.

11 Calculate an estimate for the area under the graph of \( y = 2^x \) between \( x = 0 \) and \( x = 2 \) using the trapezoidal rule and:
   a) 2 equal intervals
   b) 4 equal intervals.

12 Using the trapezoidal rule with:
   a) 2 equal intervals
   b) 4 equal intervals
   find the approximate area under the graph of \( y = \sqrt{\sin x} \) between \( x = 0 \) and \( x = \pi \).

---

Is there a connection between area and volume of revolution?

The area under a curve can be found directly from integration. Likewise, the volume of revolution can also be found from integration. In both cases it is assumed that the curves have functions which can be readily integrated.

This investigation examines the relative size of the area produced by a curve and the \( x \)-axis, and the volume of revolution produced. Curves will be restricted to those of the type \( f(x) = ax^n \).

1 Consider the function \( f(x) = ax^n \), where \( a = \frac{1}{2}, 1, 2 \) and \( 4 \) and \( n = 1 \). Find the area enclosed by the curve, the \( x \)-axis and the line \( x = 1 \) for each of the four values for \( a \). In what manner is the area dependent on the gradient of the line \( a \)?

2 For each of the four curves, find the volume of revolution. In what way is the volume dependent on \( a \)?

3 Consider the function \( f(x) = ax^n \), where \( n = \frac{1}{2}, 1, 2 \) and \( 4 \) and \( a = 1 \). Find the area enclosed by the curve, the \( x \)-axis and the line \( x = 1 \) for each of the four values for \( n \). In what manner is the area dependent on the value of \( n \)?

4 For each of the four curves, find the volume of revolution. In what way is the volume dependent on the value of \( n \)?

5 Finally, compare the sizes of the areas found in parts 1 and 3 to the volumes found in parts 2 and 4. In particular, investigate the ratio of the volume of revolution \( V \) to the area \( A \). In what way does the ratio \( \frac{V}{A} \) depend on the values of \( a \) and \( n \) for the general function \( f(x) = ax^n \)? What happens to the ratio as \( n \to \infty \) when \( a = 1 \)?

6 Write a brief report detailing your findings being careful to illustrate your work with graphs and with calculations.
### Common antiderivatives

- The table below lists common antiderivatives.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ax^n$</td>
<td>$\frac{ax^{n+1}}{n+1} + c$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\log_e kx + c$</td>
</tr>
<tr>
<td>$e^{kx}$</td>
<td>$\frac{e^{kx}}{k} + c$</td>
</tr>
<tr>
<td>$\sin kx$</td>
<td>$-\frac{\cos kx}{k} + c$</td>
</tr>
<tr>
<td>$\cos kx$</td>
<td>$\frac{\sin kx}{k} + c$</td>
</tr>
<tr>
<td>$\sec^2 kx$</td>
<td>$\frac{\tan kx}{k} + c$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{a^2 - x^2}}, x \in (-a, a)$</td>
<td>$\sin^{-1} \frac{x}{a} + c$</td>
</tr>
<tr>
<td>$-\frac{1}{\sqrt{a^2 - x^2}}, x \in (-a, a)$</td>
<td>$\cos^{-1} \frac{x}{a} + c$</td>
</tr>
<tr>
<td>$\frac{a}{a^2 + x^2}$</td>
<td>$\tan^{-1} \frac{x}{a} + c$</td>
</tr>
</tbody>
</table>

### Substitution where the derivative is present in the integrand

- $\int f'(x) \ [f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$
- $\int \frac{f'(x)}{f(x)} \, dx = \log_e f(x) + c$

### Linear substitution

The integral $\int f(x) \ [g(x)]^n \, dx, n \neq 0$ may be successfully antidifferentiated using the substitution $u = g(x)$, provided that $g(x)$ is linear. The function $f(x)$ must be written in terms of $y$ also.

### Useful trigonometric identities

- Trigonometric identities are used to integrate even and odd powered trigonometric functions:
  \[
  \sin^2 ax = \frac{1}{2} (1 - \cos 2ax) \\
  \cos^2 ax = \frac{1}{2} (1 + \cos 2ax) \\
  \sin ax \cos ax = \frac{1}{2} \sin 2ax
  \]
Antidifferentiation using partial fractions

Many rational expressions can be antidifferentiated by transforming the expressions into partial fractions. Two common types are shown below.

<table>
<thead>
<tr>
<th>Rational expression</th>
<th>Equivalent partial fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{f(x)}{(ax + b)(cx + d)} ) where ( f(x) ) is a linear function</td>
<td>( \frac{A}{ax + b} + \frac{B}{cx + d} )</td>
</tr>
<tr>
<td>( \frac{f(x)}{(ax + b)^2} ) where ( f(x) ) is a linear function</td>
<td>( \frac{A}{(ax + b)^2} + \frac{B}{ax + b} )</td>
</tr>
</tbody>
</table>

**Definite integrals**

- \( \int_a^b f(x) \, dx = [F(x)]_a^b \)
  - \( = F(b) - F(a) \), where \( F(x) \) is an antiderivative of \( f(x) \).
- The definite integral \( \int_a^b f(x) \, dx \) can be found only if the integrand \( f(x) \) exists for all values of \( x \) in the interval \([a, b] \); that is, \( a \leq x \leq b \).

**Areas under curves**

- \( y = f(x) \)
  - Area = \( \int_a^b f(x) \, dx \)

- \( y = g(x) \)
  - Area = \( \int_a^b g(x) \, dx \)

- \( y = f(x) \)
  - Area = \( \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \)

- \( x = f(y) \)
  - Area = \( \int_a^b f(y) \, dy + \int_c^b f(y) \, dy \)
Areas between curves

- \[
\text{Area} = \int_{a}^{c} [g(x) - f(x)] \, dx + \int_{c}^{b} [f(x) - g(x)] \, dx
\]

- \[
\text{Area} = \int_{a}^{b} [g(y) - f(y)] \, dy
\]

Volumes of solids of revolution

- About \( x \)-axis: \( V = \pi \int_{a}^{b} [f(x)]^2 \, dx \).
- About \( y \)-axis: \( V = \pi \int_{a}^{b} [f(y)]^2 \, dy \).
- Between two functions \( f(x) \) and \( g(x) \) where \( f(x) \geq g(x) \):
  \[
  V = \pi \int_{a}^{b} [f(x)]^2 - [g(x)]^2 \, dx
  \]

Approximate evaluation of definite integrals and areas

- Approximate measure of \( \int_{a}^{b} f(x) \, dx \) using the midpoint rule:
  \[
  \int_{a}^{b} f(x) \, dx = \delta x \left[ f\left( \frac{x_0 + x_1}{2} \right) + f\left( \frac{x_1 + x_2}{2} \right) + \ldots + f\left( \frac{x_{n-1} + x_n}{2} \right) \right]
  \]
  where: \( n = \) the number of intervals used
  \[
  \delta x = \frac{b-a}{n}
  \]
  \[
  x_0 = a
  \]
  \[
  x_n = b
  \]
- Approximate measure of \( \int_{a}^{b} f(x) \, dx \) using the trapezoidal rule:
  \[
  \int_{a}^{b} f(x) \, dx = \frac{\delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n) \right]
  \]
  where: \( n = \) the number of intervals used
  \[
  \delta x = \frac{b-a}{n}
  \]
  \[
  x_0 = a
  \]
  \[
  x_n = b.
  \]
Multiple choice

1. The expression \( \int (x - 1)(x^2 - 2x)^5 \, dx \) is equal to:
   
   A. \( \int u^5 \, du \)  
   B. \( \frac{1}{2} \int u^5 \, du \)  
   C. \( 2 \int u^5 \, du \)  
   D. \( 5u^4 \, du \)  
   E. \( u^4 \, du \)

2. An antiderivative of \( \frac{\sin x}{\cos^3 x} \) is:
   
   A. \( \frac{1}{\cos^4 x} \)  
   B. \( -\frac{1}{\cos^2 x} \)  
   C. \( -\frac{1}{\sin^2 x} \)  
   D. \( \frac{1}{2 \cos^2 x} \)  
   E. \( \frac{1}{4 \cos^2 x} \)

3. The expression \( \int 6 \sec^2 x \tan^4 x \, dx \) is equal to:
   
   A. \( 2 \int \sec^4 x \, dx \)  
   B. \( \int \sec^4 x \, dx \)  
   C. \( \frac{1}{2} \int \sec^4 x \, dx \)  
   D. \( \int \sec^2 x \, dx \)  
   E. \( 2 \int \sec^2 x \, dx \)

4. The antiderivative of \( x(x + 2)^{10} \) is:
   
   A. \( x^{11} + c \)  
   B. \( (x + 2)^{11} + c \)  
   C. \( \frac{(x + 2)^{11}(11x - 2)}{132} + c \)  
   D. \( \frac{(x + 2)^{11}(12x - 11)}{132} + c \)  
   E. \( x(x + 2)^{11} + c \)

5. \( \int x\sqrt{2 - x} \, dx \) is equal to:
   
   A. \( \frac{2}{5}(2 - x)^{\frac{5}{2}} - 2(2 - x)^{\frac{3}{2}} + c \)  
   B. \( \frac{5}{2}(2 - x)^{\frac{5}{2}} - 3(2 - x)^{\frac{3}{2}} + c \)  
   C. \( x^2(2 - x)^{\frac{3}{2}} + c \)  
   D. \( \frac{1}{3}(2 - x)^{\frac{5}{2}} + 2(2 - x)^{\frac{3}{2}} + c \)  
   E. \( -\frac{7}{15}(2 - x)^{\frac{3}{2}}(3x + 4) + c \)

6. Using an appropriate substitution, \( \int e^{2x}\sqrt{e^x - 1} \, dx \) is equal to:
   
   A. \( \int (u^3 + u^\frac{1}{2}) \, du \)  
   B. \( \int (2u^3 + u^\frac{1}{2}) \, du \)  
   C. \( \int (u^5 + 2u^3 + u^\frac{1}{2}) \, du \)  
   D. \( \int (u^2 + 2u^\frac{3}{2}) \, du \)  
   E. \( \int (u^\frac{5}{2} + u^\frac{1}{2}) \, du \)

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**CHAPTER REVIEW**
7 Using an appropriate substitution, \( \int \cos^3 x \sin^2 x \, dx \) is equal to:

A \( \int (u^4 - u^2) \, du \)  
B \( \int u^2 \cos x \, dx \)  
C \( \int (u^5 - u^3) \, du \)  
D \( \int (u^3 - u^5) \, du \)  
E \( \int (u^2 - u^4) \, du \)

8 If \( f'(x) = 4 \sin^2 x \) and \( f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} \), then \( f(x) \) is equal to:

A \( 4 \cos^2 x + \frac{\pi}{2} - 2 \)  
B \( 2x - 1 + \sin 2x \)  
C \( 2x + \cos 2x \)  
D \( 2x - \cos 2x \)  
E \( 2x - \sin 2x \)

9 Using the appropriate substitution, \( \int \sin^5 x \, dx \) is equal to:

A \( \int (u^2 - u^4) \, du \)  
B \( \int (u^4 - 2u^2) \, du \)  
C \( \int (2u^2 - 1 - u^4) \, du \)  
D \( \int (-u^4 - 1) \, du \)  
E \( \int (u^4 - u^2 + 1) \, du \)

10 The expression \( \int (2 + \tan^2 x) \, dx \) is equal to:

A \( x + \sec^2 x + c \)  
B \( 2x + \sec^2 x + c \)  
C \( \tan x + c \)  
D \( x + \tan x + c \)  
E \( x \tan x + c \)

11 Given that \( \frac{1}{x^2 - 9x + 20} = \frac{1}{x - 5} - \frac{1}{x - 4}, \) \( x > 5 \), an antiderivative of \( \frac{1}{x^2 - 9x + 20} \) is:

A \( \log_e \left(\frac{x - 5}{x - 4}\right) \)  
B \( \log_e \left(\frac{x - 4}{x - 5}\right) \)  
C \( \log_e (x^2 - 9x + 20) \)  
D \( \log_e (x - 5) - \frac{1}{(x - 4)^2} \)  
E \( \frac{1}{(x - 5)^2} - \frac{1}{(x - 4)^2} \)

12 The expression \( \int \frac{2x + 3}{(x + 1)^2} \, dx \) is equal to:

A \( \frac{-2}{x + 1} + c \)  
B \( \frac{-2}{(x + 1)^2} + 3 \log_e (x + 1) + c \)  
C \( \log_e (x + 1) + \frac{1}{x + 1} + c \)  
D \( 2 \log_e (x + 1) - \frac{1}{x + 1} + c \)  
E \( \log_e(x + 1)^2 + c \)

13 The integral \( \int_a^b \frac{1}{\sqrt{9 - x^2}} \, dx \) can be evaluated over the largest domain of:

A \( (-9, 9) \)  
B \([-3, 3]\)  
C \( (-3, 0) \)  
D \( R \)  
E \( (-3, 3) \)
14 The value of \( \int_{-1}^{1} \frac{x^2}{x^3 + 1} \, dx \) is:
   A \( \log_e 2 \)  
   B \( \frac{1}{3} \log_e 2 \)  
   C \( 3 \log_e 2 \)  
   D \( \log_e 4 \)  
   E unable to be calculated.

15 The expression \( \int_{0}^{\pi} 2x \cos x^2 \, dx \) is equal to:
   A \( \cos \pi \)  
   B \( \sin (\pi^2) \)  
   C 0  
   D 1  
   E 2

16 The integral representing the shaded area of this curve is equal to:
   A \(2 \times \int_{0}^{1} (x^2 - 1) \, dx\)  
   B \(\int_{-1}^{1} (x^2 - 1) \, dx\)  
   C \(2 \times \int_{1}^{0} (x^2 - 1) \, dx\)  
   D \(\int_{0}^{1} (1 - x^2) \, dx\)  
   E \(\int_{0}^{2} (x^2 - 1) \, dx\)

17 The area between the curve \( y = \sin x \) and the line \( y = x \) from \( x = 0 \) to \( x = 1 \) (see diagram) is approximately equal to:
   \[ \int_{0}^{1} (x - \sin x) \, dx \]
   A 0.04 square units  
   B 1.04 square units  
   C 0.54 square units  
   D 0.84 square units  
   E 0.34 square units

18 The shaded area (in square units) on the graph below is equal to:
   \[ \int_{0}^{4} (x - 2)^2 \, dx \]
   A \( \frac{16}{3} \)  
   B 16  
   C \( \frac{32}{3} \)  
   D \( \frac{8}{3} \)  
   E 8
Questions 19 and 20 refer to the shaded area in the figure below.

19. The volume generated when the region is rotated about the x-axis is equal to:
   - A $\pi \int_{0}^{1} (4 - 2x^2 + x^4 - x) \, dx$
   - B $\pi \int_{0}^{1} (4 + x^4 - x) \, dx$
   - C $\pi \int_{0}^{1} (4 - 3x^2 + x^4) \, dx$
   - D $\pi \int_{0}^{1} (4 - 4x^2 + x^4 + x) \, dx$
   - E $\pi \int_{0}^{1} (2 - x^2 - \sqrt{x}) \, dx$

20. The volume generated when the region is rotated about the y-axis is equal to:
   - A $\pi \int_{0}^{2} (2 - y) \, dy$
   - B $\pi \int_{0}^{2} (2 - y) \, dy + \pi \int_{1}^{2} y^2 \, dy$
   - C $\pi \int_{0}^{2} y^2 \, dy$
   - D $\pi \int_{0}^{2} y^2 \, dy + \pi \int_{1}^{2} (2 - y) \, dy$
   - E $\pi \int_{0}^{2} (2 - y - y^2) \, dy$

21. The approximate value of $\int_{1}^{4} \frac{e^x}{x} \, dx$ using the trapezoidal rule and 3 equal intervals is:
   - A $\frac{1}{12} (6e + 3e^2 + 2e^3 + e^4)$
   - B $\frac{1}{8} (4e + 4e^2 + 2e^3 + e^4)$
   - C $\frac{1}{12} (6e + 6e^2 + 4e^3 + e^4)$
   - D $\frac{1}{4} (2e + 2e^2 + e^3 + \frac{1}{2} e^4)$
   - E $\frac{1}{24} (12e + 12e^2 + 8e^3 + 3e^4)$

22. The approximate value of the area under the curve $y = x^2 + 1$ from $x = -1$ to $x = 1$ (using the midpoint rule with four equal intervals) is:
   - A 2.625 square units
   - B 1.3125 square units
   - C 2.5 square units
   - D 2.75 square units
   - E 1.95 square units

Short answer

1. Find the antiderivative of:
   - a $(\cos x) e^{\sin x}$
   - b $\frac{(\log x)^2}{x}$

2. Find the indefinite integral $\int \frac{x}{\sqrt{x + 1}} \, dx$.

3. Find:
   - a $\int \cos^2 2x \, dx$
   - b $\int \frac{\sin^2 x}{4} \cos^2 \frac{x}{4} \, dx$
4 Find an antiderivative of \( f(x) \) where \( f(x) = \frac{x^2 - 2x - 12}{x^2 - 7x - 8} \).

5 Find \( f(x) \) if \( f'(x) = \sin 2x \cos x \) and \( f(\pi) = 1 \).

6 If \( f'(x) = 2\sqrt{1-x^2} \) and \( f(0) = -3 \), find \( f(x) \).
   \( \text{(Hint: Use the substitution} x = \sin \theta \text{ to antidifferentiate.)} \)

7 Evaluate:
   \[
   \begin{align*}
   \text{a} & \quad \int_{0}^{2} \frac{1}{4 + x^2} \, dx & \quad \text{b} & \quad \int_{-2}^{1} \frac{x}{\sqrt{2-x}} \, dx
   \end{align*}
   \]

8 a Sketch a graph which shows the region enclosed by the curve \( y = \log_e x \), the line \( y = 2 \) and the \( x \)- and \( y \)-axes.

   b Find the area of this region.

9 What is the area bounded by the curve \( y = x^2 + 2 \) and the line \( y = 5x - 4 \)?

10 Find the volume generated when the area under the graph of \( y = e^x \), between \( x = -1 \) and \( x = 0 \), is rotated about the \( x \)-axis.

11 Find the volume of water in a hemispherical bowl of radius 8 cm if the depth is 3 cm.

12 a Find an approximate value of \( \int_{0}^{4} 2x^2 \, dx \) using four equal intervals and:
   \[
   \begin{align*}
   \text{i} & \quad \text{the midpoint rule} \\
   \text{ii} & \quad \text{the trapezoidal rule.}
   \end{align*}
   \]

   b Which result is closest to the exact answer?

**Analysis**

1 a Find the area of the shaded region on the graph at right.

   b What is the volume generated when this region is rotated about the \( x \)-axis?

   c If the region is rotated about the \( y \)-axis, find the approximate volume of the solid generated using the midpoint rule and four equal intervals. (Give your answer correct to 4 decimal places.)

![Graph of y = tan x](image)
2 The side view of the right side of a wine glass vessel can be modelled by two curves which join at $x = e$:

\[ y = 2 \log_e x, \quad 0 < x \leq e \] (red curve)
\[ y = x^2 - 2ex + e^2 + c, \quad e \leq x \leq 5 \] (blue curve)

(All measurements are in centimetres.)

a Show that the value of $c$ is 2 and find the height of the vessel correct to 2 decimal places.

b Find the volume of wine in the glass when the depth is 2 cm.

c What is the maximum volume of wine that the glass can hold (using maximum height to the nearest mm)?

3 A below-ground skating ramp is to be modelled by the curve

\[ y = \frac{2}{\sqrt{36 - x^2}}, \quad -5.98 \leq x \leq 5.98 \]

This is shown above, where the line $y = 4.086$ represents ground level. (All measurements are in metres.)

(Give all answers correct to 2 decimal places.)

a Find the maximum depth of the ramp.

b Find the area under the curve.

c Find the volume generated if this area is rotated about the $x$-axis.

d If the ramp is 20 metres long, what is the volume of dirt which must be removed?